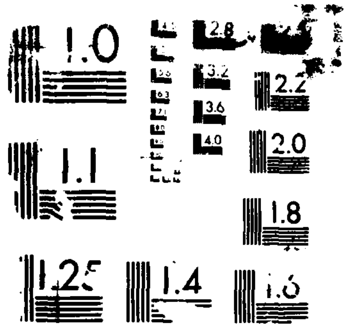
AD-A189 265 CONFORMAL GEOMETRY HOTINE'S CONJECTURE AND DIFFERENTIAL GEODESY..(U) NEW MEXICO STATE UNIV LAS CRUCES DEPT OF MATHEMATICAL SCIENCE.. J D ZUND ET AL. 27 JUL 87 1/2

UNCLASSIFIED SCIENTIFIC-1 AFGL-TR-87-0233 F/G 8/5 NL

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# Conformal Geometry, Hotine's Conjecture, and Differential Geodesy

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27 July 1987

Scientific Report No. 1

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Unclassified
SECURITY CLASSIFICATION OF THIS PAGE

# REPORT DOCUMENTATION PAGE

| | |
|---|---|
| 1a REPORT SECURITY CLASSIFICATION<br>Unclassified | 1b RESTRICTIVE MARKINGS |
| 2a SECURITY CLASSIFICATION AUTHORITY | 3 DISTRIBUTION/AVAILABILITY OF REPORT<br>Approved for public release;<br>Distribution unlimited |
| 2b DECLASSIFICATION/DOWNGRADING SCHEDULE | |
| 4 PERFORMING ORGANIZATION REPORT NUMBER(S) | 5 MONITORING ORGANIZATION REPORT NUMBER(S)<br>AFGL-TR-87-0233 |
| 6a NAME OF PERFORMING ORGANIZATION<br>New Mexico State University<br>6b OFFICE SYMBOL (If applicable) | 7a NAME OF MONITORING ORGANIZATION<br>Air Force Geophysics Laboratory |
| 6c ADDRESS (City, State, and ZIP Code)<br>Dept of Mathematical Sciences<br>Las Cruces, NM 88003 | 7b ADDRESS (City, State, and ZIP Code)<br>Hanscom AFB<br>Massachusetts 01731 |
| 8a NAME OF FUNDING/SPONSORING ORGANIZATION<br>8b OFFICE SYMBOL (If applicable) | 9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER<br>F19628-86-K-0028 |
| 8c ADDRESS (City, State, and ZIP Code) | 10 SOURCE OF FUNDING NUMBERS<br>PROGRAM ELEMENT NO: 61102F<br>PROJECT NO: 2309<br>TASK NO: G1<br>WORK UNIT ACCESSION NO: BQ |

11 TITLE (Include Security Classification)
Conformal Geometry, Hotine's Conjecture, and Differential Geodesy

12 PERSONAL AUTHOR(S)
Joseph D. Zund, Wayne A. Moore

| 13a TYPE OF REPORT | 13b TIME COVERED | 14 DATE OF REPORT (Year, Month, Day) | 15 PAGE COUNT |
|---|---|---|---|
| Scientific No. 1 | FROM ______ TO ______ | 1987 July 27 | 128 |

16 SUPPLEMENTARY NOTATION

| 17 COSATI CODES | | | 18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number) |
|---|---|---|---|
| FIELD | GROUP | SUB-GROUP | Differential Geodesy; Conformal Geometry, Hotine's Conjecture; Tensor Analysis |

19 ABSTRACT (Continue on reverse if necessary and identify by block number)


This report is based on the Ph.D. dissertation of Wayne Moore, together with an Introduction (which serves as a guide to the dissertation) and a Supplement describing details of our recent resolution of Hotine's Conjecture. In effect, it is a more comprehensive version of a paper of ours which will appear this year in Bulletin Géodésique. It includes a detailed development of conformal geometry which is more complete than that given in Chapter 10 of Hotine's Mathematical Geodesy (1969), and numerous technical aspects of Hotine's work on his conjecture which were omitted in our paper. It is hoped that this report will be helpful to mathematical geodesists who wish to further explore the contributions of Hotine and Marussi to differential geodesy.


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| 20 DISTRIBUTION/AVAILABILITY OF ABSTRACT<br>☐ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS | 21 ABSTRACT SECURITY CLASSIFICATION<br>Unclassified |
| 22a NAME OF RESPONSIBLE INDIVIDUAL<br>Christopher Jekeli | 22b TELEPHONE (Include Area Code) (617) 377-5255<br>22c OFFICE SYMBOL AFGL/LWG |

DD FORM 1473, 84 MAR  83 APR edition may be used until exhausted. All other editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE
Unclassified

CONTENTS

SUPPLEMENT (by J.D. ZUND)

## INTRODUCTION (by J.D. ZUND)

The major part of this research report is essentially the Ph.D. dissertation of Wayne Moore, "Conformal Geometry and Differential Geodesy," which was written under my direction at New Mexico State University with the support of the Air Force Geophysics Laboratory. This dissertation is reproduced in its original form, apart from the correction of minor typographical errors. This report is prepared in the hope that it will be useful to mathematical geodesists who are interested in further exploring the fascinating ideas of Martin Hotine. In addition to the dissertation, I have prepared a supplement which describes joint work of Dr. Moore and myself which was done after completion of the dissertation. The contents of the dissertation and supplement definitely settle Hotine's conjecture on the use of triply-orthogonal systems of surfaces as a natural coordinate system in differential geodesy. Hotine made his conjecture in 1966, and it is included in his treatise, HOTINE [1969][1], but was unresolved until our investigation. We show that this conjecture is false, however, we believe that this negative result in no way impairs either Hotine's approach or the importance of his conception of a unified approach to three-dimensional geodesy using the notions of tensor analysis and differential geometry.

The following discussion gives a guide and commentary to the contents of this research report.

The dissertation consists of four chapters, a bibliography, and an appendix. Chapter I -- Introduction -- contains some preliminary comments on geodesy intended for mathematicians and physicists. It is not comprehensive, but is merely intended to indicate the close relationship which existed between geodesy and mathematics and physics before the twentieth century. In

[1] In this report, the cited references are those given in the dissertation.

effect, it sets the stage for the topic to be investigated in the dissertation. The chapter concludes with biographical material on Antonio Marussi and Martin Hotine, and their proposal to create a three-dimensional geodesy in the discipline which we call differential geodesy.

Chapter II -- Conformal Geometry -- is a self-contained introduction and systematic exposition of conformal geometry, i.e., the geometry of a pair of N-dimensional Riemannian spaces $V_N$ and $\hat{V}_N$ which are related by a conformal transformation. It presumes a prior acquaintance with tensor analysis, and can be regarded as supplement to the discussion in HOTINE [1969] (Chapter 10, pages 55-62) and EISENHART [1949] (Chapter II, pages 89-95). The latter is the standard reference in English on the topic, however, it is very incomplete and really merely a partial summary of known results. Chapter II begins with a brief excursion into the modern formulation of conformal geometry in Section §2. This is admittedly somewhat abstract, however, it is necessary to make precise the notion of a conformal transformation, i.e., mapping, between $V_N$ and $\hat{V}_N$. This notion, and what is meant by adopting the same local coordinate system on both $V_N$ and $\hat{V}_N$, is obscurely done in the classical literature, e.g., EISENHART [1949]. Sections §3-5 then contain a systematic development of conformal geometry using the classical formulation with particular attention being given to the dimensions $N = 2$ and $N = 3$ which are of primary importance in non-relativistic differential geodesy. The most interesting mathematical aspects of conformal geometry occur when $N \geq 4$, and the cases $N = 2$ and $N = 3$ are often inadequately treated in the literature. Section §3 introduces the basic ideas, concepts, and tensor-theoretic quantities encountered in conformal geometry, and Section §4 discusses their behavior when $N = 2$ and $N = 3$. Section §5 considers integrability conditions and conformally flat spaces. The presentation of this material in the usual references, i.e., EISENHART [1949] and SCHOUTEN [1954], is wholly inadequate and riddled with errors. Moreover, the original material is also confusing. The key results, which are the "X-Representation" of the curvature tensor (see pages 29-30) and Schouten's lemma (see Theorem 5.3, on page 33), were originally done in Schouten's symbolic notation which was not widely understood and which Schouten himself later abandoned. The material is quite subtle and intricate, and I regard Section §5 as being the clearest and best treatment of the subject which I have seen. The topic of conformally flat spaces is likely to be of considerable importance in future work in differential geodesy. Both HOTINE [1966a, 1966b] and MARUSSI [1985] (see pages 169-176) suggested that such spaces are important in studying the propagation of light in continuous isotropic refracting media. Virtually everything in Chapter II is necessary to understand the conceptual setting for resolving Hotine's Conjecture.

Chapter III -- Triply Orthogonal Systems -- is devoted to introducing the basic notions occurring in Hotine's Conjecture. Section §1 reviews the theory of triply orthogonal systems of surfaces and presents some elementary examples of such systems. It also includes the Dupin Theorem (Theorem 1.1, pages 39-41) and the Generalized Dupin Theorem (Theorem 1.2, pages 41-42). The former is proven in Section §1, however, the latter is more complicated, and its proof is deferred until Chapter IV. Section §2 is devoted to giving a tensor-theoretic derivation of the Cayley-Darboux equation and explicitly exhibits the general form of this equation. The material in this section has been accepted for publication, see ZUND/MOORE [1986], and will appear in the near future. It turns out that the Cayley-Darboux equation is the critical result in refuting Hotine's Conjecture. As will be shown in the supplement, Hotine's version of this equation is wrong, and this led him into believing that this equation was always identically satisfied in a flat three-dimensional Euclidean space $E_3$. Section §3 contains a statement and proof of

Liouville's Theorem (Theorem 3.1, pages 46-51). This important result is discussed (without attaching Liouville's name to it) in HOTINE [1969] (see page 56 and his footnote, and also in HOTINE [1966a, 1966b]). The theorem is rather deep and delineates the possible types of conformal transformations between N-dimensional Euclidean spaces $E_N$ and $\hat{E}_N$. Despite its importance, this result is rarely proven in the literature, and Section §3 concludes with a self-contained proof which is based on that given in BIANCHI [1910]. A general N-dimensional argument is given in DUBROVIN/FOMENKO/NOVIKOV [1984], but it is incomplete and is not changed in the second, 1986, Russian edition of this book.

Chapter IV -- Differential Geodesy -- presents the analytical apparatus for analyzing the Hotine Conjecture and the flaws in his alleged proof. After formally stating his conjecture in Section §1 and why he hoped it would be true, in Section §2 the question of isometric immersion of a surface $V_2$ in a $V_3$ is discussed together with the behavior of $V_2$ under a conformal mapping. Section §3 introduces the formalism of Ricci rotation coefficients and indicates how they are affected by a conformal mapping of $V_3$ into $\hat{V}_3$. Although in HOTINE [1969] this kind of reference system -- it is convenient to call it a *triad* -- was implicitly employed, this important topic was not explicitly utilized. This was a serious omission in Hotine's analysis, since, as we will show in the supplement, it will allow us to conclusively demonstrate that his Cayley-Darboux equation is wrong and not an identity! We believe that this triad formalism is particularly suited to the requirements of differential geodesy and will prove to be an important part of future developments in mathematical geodesy. Section §4 uses this formalism to concisely establish two important criteria: the Schouten-Eisenhart Theorem (Theorem 4.1, pages 59-61) and the Ricci-Finzi Theorem (Theorem 4.2, pages 61-63), for the conformal flatness of a $V_N$. Theorem 4.3, (pages 63-64) furnishes a new Schouten-Eisenhartlike criterion for a $V_3$. A more abstract version of the results of Section §4 has been submitted to Tensor N.S. for publication. The properties of congruences of curves being normal, geodesic, and canonical are studied in Section §5 using rotation coefficients and a proof of the Generalized Dupin Theorem is given (pages 70-71). This new proof has been accepted for publication, see MOORE/ZUND [1986]. An example of how to find a canonical congruence is presented in the Appendix (pages 101-103). Section §6 contains a rotation coefficient formulation of the Cayley-Darboux equation. Section §7 now begins our analysis of Hotine's Conjecture by outlining the steps of his argument. Section §8 shows by using Theorem 8.1 (page 81) that part of Hotine's proof is false. Section §9 examines when the conformal image of a normal congruence of curves can be a geodesic normal congruence and suggests that another step of Hotine's argument is shaky. Section §10 briefly considers the question of a conformal mapping between a pair of surfaces $V_2$ and $\hat{V}_2$, and shows that a formula for the conformal image of the geodesic curvature given in MARUSSI [1985] (see his page 150) is incorrect. Finally, in Section §11 our results are applied to give a critique of Hotine's argument. It is shown that further flaws occur in his procedure: he specialized the conformal function defining his mapping and made essential use of coordinates/equations which are valid only at a point of a $V_2$. The effect of either of these mistakes is to demand that his (curved) surfaces degenerate into (flat) planes. None of these steps can be easily rectified, and when, taken together, they strongly suggest that Hotine's argument is fatally flawed, and that his conjecture is false.

The Supplement (written by J.D. Zund) now furnishes the conclusive reasons why Hotine's conjecture is false. It is intended as a companion to our joint paper, "Hotine's Conjecture in Differential Geodesy," which will be published in Bulletin Géodésique. In this supplement we not only extend the material in Dr. Moore's dissertation, but provide the details of our refutation of the Hotine Conjecture which had to be omitted in our paper. After some introductory comments in Section §1, we translate Hotine's Cayley-Darboux equation into the rotation coefficient formalism and prove that it is not equivalent to the true Cayley-Darboux equation in Section §2. This shows that the Cayley-Darboux equation is not an identity as Hotine claimed, and hence his conjecture cannot be true. Finally, in Section §3 we discuss the physical consequences of this result, and what they mean for differential geodesy.

In conclusion, Dr. Moore and I would like to express our gratitude to the Air Force Geophysics Laboratory and in particular to Dr. Jekeli for his invitation to prepare this research report. The cooperation and support of this research under contract F 19628-86-K-0028 "Conformal and Non-Conformal Transformations in Differential Geodesy" is gratefully acknowledged.

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July 1987

# CONFORMAL GEOMETRY AND DIFFERENTIAL GEODESY

BY

WAYNE ANTHONY MOORE, B.S., M.S.

A Dissertation submitted to the Graduate School
in partial fulfillment of the requirements
for the Degree
Doctor of Philosophy

Major Subject: Mathematics (Applied)
Related Area: Mathematics (Pure)

New Mexico State University
Las Cruces, New Mexico
December 1986

# CHAPTER I

# INTRODUCTION

## §1. GEODESY AND MATHEMATICS

Theoretical geodesy is the study of the size and shape of the Earth. This is not geographic shape, but rather the shape of the surface of mean sea level and its continuation under the earth's crust. This surface of mean sea level is called a geoid and by definition it is an equipotential surface of the earth's gravitational field.

The history of geodesy may be divided into three eras: the spherical era, the spheroidal era, and the geoidal era. The spherical era dates from the time of the Greeks to approximately 1670. The spheroidal era stretches from 1670 to approximately 1830. The geoidal era spans from 1830 to the present. An eminent mathematician was pivotal in the transition between each of these eras.

Pythagoras is normally credited as being first to conceive of the earth as spherical. He reasoned that since the sun and moon are spherical, then the earth must have a similar shape. Aristotle made what was probably the first scientific estimate of the size of the earth. However, since we have no idea of the size of his unit of length (the stadium) it is impossible to estimate the accuracy of his work. In the third century B.C., Erastosthenes, considered one of the founders of geodesy, devised the idea of measuring the size of the great circle arc between the North and South poles. His technique was based on measuring the difference in the angle of the sun's rays at Alexandria and Syrene (now Aswan), which he assumed to be on the same longitude. Although Erastosthenes' estimate is 15% too large when compared with modern estimates, his idea is very modern in spirit and the error is due to the crudeness of his measurements. In the first century B.C., Posidonius made another estimate of the size of the earth based on the difference in the angle the star Canopus made with the horizon at Alexandria and Rhodes. His estimate was also 11% too large.

After Posidonius interest in geodesy lapsed for some 1500 years when it was rekindled by the need for accurate maps and the exploration of the New World. In 1617, Snell used triangulation as a method to determine distances. This was a breakthrough since this method was much more accurate than direct measurement. In 1669, Picard used a telescope to determine latitude as well as in triangulation. Picard's work is extremely important since Newton used his estimate of length of the arc of a degree of latitude to show that gravity extends beyond the surface of the earth and determines the motion of the moon. This estimate of the length of a degree of latitude ultimately became the basis of the metric system.

The period between Erastosthenes and Picard is the spherical era of geodesy. The work of Isaac Newton inaugurates the spheroidal era. Newton's discoveries in mechanics, i.e., laws of motion, and the formulation of the law of gravitational attraction, are crucial steps in determining the shape of the earth. Newton used a theoretical argument based on the hydrostatic equilibrium of the oceans to show that the earth was an oblate sphere with the major axis 1/230 longer than the minor axis. The modern value for this eccentricity is approximately 1/290. After Newton published this estimate, many people were anxious to confirm or refute it. It was reasoned that if the earth were flattened at the poles, a degree of latitude would be shorter near the poles than at the equator. The Cassinis, a family of astronomers, using estimates for a degree of latitude in the north and the south of France, determined that the earth was a <u>prolate</u> sphere and <u>not</u> an oblate sphere. This announcement naturally aroused a good deal of controversy, and in the 1730's the French Academy of Sciences sent two expeditions -- one to Lapland and one to Peru -- to settle the matter. Their measurements showed that the earth was indeed oblate with eccentricity 1/178. It is interesting that Newton, sitting in his room in Cambridge, could produce a better estimate than the French Academy could with two expeditions.

With the initiation of mechanics and gravitation by Newton other mathematicians were fast to extend his studies. Euler developed the mechanics of rigid bodies, Lagrange analytical mechanics, Legendre potential theory, Laplace the mechanics of rotating fluid masses. A powerful mathematician, Clairaut, computed the variation of gravity with latitude. Daniel Bernoulli and Laplace studied tides and methods to predict them. This research was not "pure" research, which was then applied to physical problems, but mathematics that was developed to understand physical and geodetic phenomena.

As Newton opened the spheroidal era, it was Gauss (at least in spirit) who started the geoidal era. Gauss became geodetic consultant for the Prussian army in 1799 and in 1820 was involved in field work where he developed new instruments for surveying. He adapted his method of least squares to geodetic measurements and developed the Gaussian probability distribution to smooth observational errors. Using data from geodetic measurements, he was led to develop his theory of curved surfaces. Gauss's intrinsic geometry of surfaces eventually led Riemann to the general theory of intrinsic geometry.

Mathematicians and physicists who followed Gauss developed and extended his work. Green further developed potential theory (he coined the term). Stokes calculated the undulations of the geoid from the theoretical ellipsoid (Stokes Theorem). Rayleigh and Poincaré extensively studied tides.

Starting around 1900 the close connection between geodesy and mathematics begins to diminish. This was caused by exciting new problems in mathematics causing it to become divorced from physics. Moreover, physics was evolving into modern physics and leaving behind classical physics with which goedesy was primarily concerned. Moreover, with the development of accurate instrumentation geodesy became less concerned with mathematical methods and theoretical physics.

## §2. HOTINE, MARUSSI AND THREE-DIMENSIONAL GEODESY

The technical achievements of the 1950's made it possible for geodesy to at last leave the surface of the earth. High flying aircraft and satellites could measure gravity high above the earth and submarines could measure it deep below the sea. The geodetic community was slow to take advantage of these new technologies as MARUSSI[1985], p. 6, notes:

> The third dimension thus appears, in practical Geodesy, as an intruder in the flourishing paradise of the Geodesist, which comprises the surfaces of the ellipsoid and the geoid; and when this intruder comes to claim his proper rights, no efforts are spared to get rid of him as quickly as possible, and with the least trouble, by means of a weighty battery of corrections and reductions, which for two centuries has figured in every treatise on Geodesy.

The first two advocates of three-dimensional geodesy were Antonio Marussi and Martin Hotine. Their advocacy was for a point of view -- that geodesy is inherently three-dimensional. Both men began to apply and develop mathematics so that this viewpoint could be practically put to use.

Antonio Marussi was born in Trieste, Italy in 1908. He received a Ph.D. degree in mathematics at the University of Bologna and then joined the Istituto Geografico Militare (Italian geodetic and mapping agency) in Florence. During his twenty years with them he modernized geodetic procedures, adopted international standards, and streamlined computing schemes. In 1952, he accepted a professorship at the University of Trieste.

Marussi introduced advanced mathematical tools such as tensor analysis to geodesy. He also recognized the practical aspects of geodesy by doing field work in numerous countries. He died in 1984.

Martin Hotine was a career military officer and rose to the rank of Brigadier. He attended the Royal Military Academy at Woolwich, England and was commissioned into the Royal Engineers in 1917. During his career in the British Army he pioneered practical methods of topographic mapping using aerial photography, did geodetic surveying in East Africa, served with the British Ordinance Survey, and during the Second World War he served as Deputy Director of Survey in the British Expeditionary Force. Hotine earned a degree in engineering from Magdalene College, Cambridge, during his time in the service.

Hotine retired from the military in 1946 and became the first director of the Directorate of Overseas Surveys where he served until 1963. He then jointed the U.S. Coast and Geodetic Survey. When the Environmental Sciences Services Administration was formed, he became a research staff member of this agency's laboratory at Boulder, Colorado. It was during this time that he wrote HOTINE[1969] and pioneered the systematic use of tensor techniques in geodesy. Hotine died in 1968.

Hotine and Marussi came to geodesy from vastly different directions. Both men achieved distinction in theoretical and practical geodesy and came to similar conclusions concerning theoretical work. Their viewpoint is summed up by MARUSSI[1985]:

> *In effect, what we know today of the earth's gravity field owes very much more to work done at the desk, with pencil and paper, than to observations made with instruments in nature,* (p. 5).

## §3. NOTES ON THIS DISSERTATION

In this dissertation we have examined a conjecture by Martin Hotine on triply orthogonal coordinate systems. In Chapter II, we introduce notation and develop machinery for conformal geometry. In Chapter III, triply orthogonal coordinate systems are discussed. Chapter IV then deals with Hotine's conjecture.

We have employed a somewhat unusual method of listing items in the bibliography. This method is based on that employed by J.A. Schouten in his book SCHOUTEN[1954], which is probably the most comprehensive book on tensor analysis. Items in the bibliography are listed by author's name and listed chronologically. References to specific items are indicated by the author's name in capital letters with the date displayed in brackets. In case of several items in a given year, the dates have lower case Latin indices attached. General references to an author are given by citing the author's name without complete capitalization. We are grateful to Mary Eberhardt of the Graduate School for her understanding and advice on how best to use Schouten's system in our bibliography.

We have used the Einstein summation convention on repeated indices. When we do not want to sum on repeated indices the letters "NS" for "no sum" will be written by the equation.

## CHAPTER II

## CONFORMAL GEOMETRY

### §1. INTRODUCTION AND BASIC NOTATION

The purpose of this Chapter is three-fold:

(i) to introduce the notation and conventions to be employed,

(ii) to relate the classical and modern terminology,

(iii) to provide proofs of some "well known" classical results which are not easily accessible.

Our notation is essentially that of EISENHART[1949], with slight modifications. However, our presentation is considerably more detailed. Let $V_N$ denote an N-dimensional Riemannian manifold. Initially we assume that $N > 3$ and the metric tensor is positive definite. The cases $N = 2$ and $N = 3$ will be dicussed separately. The symbolism ":=" and "≡" will denote "equal by definition" and "identity", respectively. Free indices will be $h, i, j, k, \ell, m, n$ and summed indices (used for emphasis when only some indices are summed) will be $p, q, r, s, t$. In the case $N = 2$ or 3 no special indices will be used. Later, special conventions will be made, e.g., when $N = 2$ Greek indices will be used.

Section 2 uses its own notation, which will be introduced there. Additional notation will be given where appropriate.

### §2. THE MODERN FORMULATION

Definition (2.1). Let $M$ and $\hat{M}$ be smooth N-dimensional Riemannian manifolds (all manifolds are assumed connected and of class $C^{\infty}$). The Riemannian manifolds $M$ and $\hat{M}$ are **locally conformal**, whenever,

(i) for every $p \in M$ and $\hat{p} \in \hat{M}$ there exists a neighborhood of $p : U \subseteq M$ and a neighborhood of $\hat{p} : \hat{U} \subset \hat{M}$, and a map $f : U \to \hat{U}$, $f(U) = \hat{U}$ such that $f$ is a diffeomorphism with $f(p) = \hat{p}$;

(ii) there exists a $C^{\infty}$ function $\lambda : U \to \mathbb{R}$ such that for any pair of tangent vectors $X_p$ and $Y_p$ in the tangent space of $M$ at $p$, $T_p(M)$, that satisfies

(a) $\lambda > 0$ on $U$,

(b) for the inner products $\langle\ ,\ \rangle_M$ and $\langle\ ,\ \rangle_{\hat{M}}$ in the respective tangent spaces $T_p(M)$ and $T_{\hat{p}}(\hat{M})$,

one has

$$\langle f_* X_p, f_* Y_p \rangle_{\hat{M}} = \lambda(p) \langle X_p, Y_p \rangle_M , \tag{2.1}$$

where $f_* : T_p(M) \to T_{\hat{p}}(\hat{M})$ is the derivative (or Jacobian) map of $f$. The map $f$ is a local conformal map, and the function $\lambda$ is called the **conformal factor**.

The derivative map, $f_*$, is defined by

$$(f_* X_p)g := X_p(g \circ f)$$

where $g \in C^{\infty}(M, \mathbb{R})$.

The local charts $\{U,h\}$ and $\{\hat{U},\hat{h}\}$ of $M$ and $\hat{M}$, respectively, are related by the commutative diagram

$$(2.2)\qquad \begin{tikzcd} U \subseteq M \arrow[rr, "f"] \arrow[dr, "h"'] & & \hat{U} \subseteq \hat{M} \arrow[dl, "\hat{h} := h\circ f^{-1}"] \\ & \mathbb{R}^N & \end{tikzcd}$$

and the local coordinate systems $\{x^i\}$ and $\{\hat{x}^i\}$ $(i = 1, \ldots, N)$ of $h(p)$ and $\hat{h}(\hat{p})$ in $U$ and $\hat{U}$, respectively, are related by the commutative diagram

$$(2.3)\qquad \begin{tikzcd} U \arrow[r, "f"] \arrow[d, "h"'] & \hat{U} \arrow[d, "\hat{h}"] \\ \mathbb{R}^N \arrow[r, "\tilde{f}"] & \mathbb{R}^N \end{tikzcd},$$

i.e., $\hat{x}^i = \tilde{f}(x^i)$, where $\tilde{f}$ is the identity map. This fact, although obvious from (2.2) and (2.3), is often obscured in the classical literature. This is the meaning of the expression "imposing the same coordinates on both manifolds". See HOTINE[1969], p. 55.

To exhibit the familiar tensor expressions for (2.1), we take a natural coordinate basis $\underset{\sim}{e}_i := \frac{\partial}{\partial x^i}$ in $T_p(M)$, and denote the map $\lambda : p \to \mathbb{R}$ by $\lambda(p) = e^{2\sigma}$ where $\sigma = \sigma(x^i)$. The derivative mapping acts on the basis vectors $\underset{\sim}{e}_i$ according to

$$(2.4)\qquad f_*(\underset{\sim}{e}_i) = f_*\left(\frac{\partial}{\partial x^i}\right) := \frac{\partial}{\partial \hat{x}^i} = e^{\sigma}\frac{\partial}{\partial x^i}\ .$$

Then since

$$(2.5)\qquad \langle \underset{\sim}{e}_i, \underset{\sim}{e}_k \rangle := g(\underset{\sim}{e}_i, \underset{\sim}{e}_k) = g_{ik}\ ,$$

and

$$(2.5a) \qquad \langle \hat{e}_i, \hat{e}_k \rangle := \hat{g}(\hat{e}_i, \hat{e}_k) \equiv \hat{g}_{ik} \; ,$$

(2.1) becomes

$$(2.6) \qquad \hat{g}_{ik} = e^{2\sigma} g_{ik} \; .$$

The tensors $g$ and $\hat{g}$ with components $g_{ik}$ and $\hat{g}_{ik}$ respectively, are the metric tensors of $M$ and $\hat{M}$, respectively. This is the usual tensor expression defining a **locally conformal map** between $M$ and $\hat{M}$ (EISENHART[1949]). Classically, e.g., in DUBROVIN/FOMENKO/NOVIKOV[1984], one often encounters the expression

$$(2.7) \qquad \hat{g}_{ij} = g_{rs} \frac{\partial x^r}{\partial \hat{x}^i} \frac{\partial x^s}{\partial \hat{x}^j}$$

which reduces to (2.6) by virtue of our definition of the map $f_*$ in (2.4).

Actually (2.6) is improperly written since it relates components of tensors defined in the tensor products of different cotangent spaces. The correct expression requires introducing the pullback (or restriction) mapping of the cotangent spaces, i.e.,

$$f^* : T^*_p(\hat{M}) \otimes \ldots \otimes T^*_p(\hat{M}) \to T_p(M) \otimes \ldots \otimes T_p(M)$$

defined by

$$(f^* a)(p)(\underset{\sim}{v}_1, \ldots, \underset{\sim}{v}_k) = a(f(p))(f_*(\underset{\sim}{v}_1), \ldots, f_*(\underset{\sim}{v}_k))$$

where $a$ is a covariant tensor of order $k$ and $\underset{\sim}{v}_i \in T_p(M)$, and requires writing

$$(2.8) \qquad f^*\hat{g} = e^{2\sigma} g ,$$

or classically,

$$(2.9) \qquad f^*\hat{g}_{ij} = e^{2\sigma} g_{ik} .$$

However, the use of (2.6) is so pervasive in the literature that in this dissertation we will always use it. We will always use the term **conformal** to mean locally conformal and henceforth write $M = V_N$ and $\hat{M} = \hat{V}_N$ as in EISENHART[1949].

We now give expressions for the Levi-Civita connections and the curvature operators on $M$ and $\hat{M}$, respectively. Writing $\hat{g} = \lambda g$, $\lambda > 0$ and let $\nabla$ and $\hat{\nabla}$ denote the Levi-Civita connections compatible with $g$ and $\hat{g}$, respectively. Then using

$$(2.10) \qquad \langle \nabla_{\underset{\sim}{X}} \underset{\sim}{Y}, \underset{\sim}{Z} \rangle = \frac{1}{2} (\underset{\sim}{X}\langle \underset{\sim}{Y}, \underset{\sim}{Z}\rangle + \underset{\sim}{Y}\langle \underset{\sim}{Z}, \underset{\sim}{X}\rangle - \underset{\sim}{Z}\langle \underset{\sim}{X}, \underset{\sim}{Y}\rangle$$
$$+ \langle \underset{\sim}{Z}, [\underset{\sim}{X}, \underset{\sim}{Y}]\rangle + \langle \underset{\sim}{Y}, [\underset{\sim}{Z}, \underset{\sim}{X}]\rangle - \langle \underset{\sim}{X}, [\underset{\sim}{Y}, \underset{\sim}{Z}]\rangle) ,$$

((2.10) is just the definition of the Christoffel symbols) we have

$$(2.11) \qquad \hat{\nabla}_{\underset{\sim}{X}} \underset{\sim}{Y} = \nabla_{\underset{\sim}{X}} \underset{\sim}{Y} + \frac{1}{2} ((\underset{\sim}{X}\psi)\underset{\sim}{Y} + (\underset{\sim}{Y}\psi)\underset{\sim}{X} - \langle \underset{\sim}{X}, \underset{\sim}{Y}\rangle \nabla\psi)$$

where $\psi := \log \sigma$ .

Calculating the curvature form for $\hat{\nabla}$ we have

$$
\begin{aligned}
(2.12)\quad \hat{R}(X,Y)Z = R(X,Y)Z &+ \frac{1}{2}\,(h_\psi(X,Z)Y - h_\psi(Y,Z)X \\
&+ \langle X,Z\rangle H_\psi Y - \langle Y,Z\rangle H_\psi X) + \frac{1}{4}\,[((Y\psi)(Z \\
&- \langle Y,Z\rangle \|\nabla\psi\|^2)X - ((X\psi)(Z\psi) - \langle X,Z\rangle\, \|\nabla\psi\|^2)Y \\
&+ ((X\psi)\langle Y,Z\rangle - (Y\psi)\langle X,Z\rangle \nabla\psi)]
\end{aligned}
$$

where $R(X,Y)$ is the curvature form of $\nabla$, $H_\psi$ is the **Hessian tensor** which is a tensor of type $(1,1)$ on $M$ with $H_\psi X := \nabla_X \nabla\psi$, and the Hessian form, $h_\psi$, is given by $h_\psi(X,Y) := \langle H_\psi X, Y\rangle$. $H_\psi$ is self-adjoint with respect to the Riemannian metric. See GROMOLL/KLINGENBERG/MEYER[1968].

Our viewpoint and analysis are always local. However, the distinction between local and global concepts is often fuzzy in the geodetic literature.

A stronger notion of conformality is that of **conformal equivalence**.

**Definition** 2.2. Let $M$ and $M'$ be N-dimensional Riemannian manifolds. Then $M$ and $M'$ are **conformally equivalent** whenever there exists a diffeomorphism

$$f : M \to M'$$

such that $f$ defines locally conformal maps on neighborhoods of $M$

and M' .

For example, consider the 2-sphere $S^2$ and the plane $E_2$. It is well-known that by stereographic projection $S^2$ is locally conformal to $E_2$; however, since $S^2$ is compact and $E_2$ is non-compact, these manifolds cannot be diffeomorphic.

## §3. THE CLASSICAL FORMULATION

The modern definition of conformal mapping was given in §2. In sections 3-5 we use the classical definition and terminology.

**Definition** 3.1. If the metric tensors of $V_N$ and $\hat{V}_N$ are related by

$$\hat{g}_{ij} = e^{2\sigma} g_{ij} \tag{3.1}$$

where $\sigma$ is a smooth function from $V_N$ to the real numbers, then $V_N$ and $\hat{V}_N$ are said to be **conformally related** or just **conformal**.

NB: The choice of the conformal factor as $e^{2\sigma}$ instead of $\sigma^2$ or $\lambda$ (used in §2) is mere convenience. The choice of $e^{2\sigma}$ is nice for differentiation and eliminates many unnecessary factors of 2 and $\frac{1}{2}$, but has no geometric significance. The function $\sigma$ in $e^{2\sigma}$ is the **conformal function**.

We now define the Christoffel symbols of $V_N$:

$$\Gamma_{ijk} := \frac{1}{2}(g_{ik|j} + g_{jk|i} - g_{ij|k}) , \tag{3.2}$$

(3.3) $$\Gamma_{ij}^{\ \ k} := g^{hk}\Gamma_{ijh} ,$$

where "|" followed by a subscript denotes partial differentiation with respect to local coordinates.

By direct calculation using (3.1), (3.2), and (3.3) we obtain the Christoffel symbols for $\hat{V}_N$ :

(3.4) $$\hat{\Gamma}_{ij}^{\ \ k} = \Gamma_{ij}^{\ \ k} + \Delta_{ij}^{\ \ k} ,$$

where

(3.5) $$\Delta_{ij}^{\ \ k} := \delta_i^{\ k}\sigma_j + \delta_j^{\ k}\sigma_i - g_{ij}g^{kp}\sigma_p ,$$

and where we have written

(3.6) $$\sigma_j := \sigma_{|j}$$

which are the components of $\nabla\sigma$ (with a tilde under it), i.e., the gradient of $\sigma$ .

NB: We add suffixes omitting a differentiation sign **only** on **scalar** quantities and only for **first** derivatives. HOTINE[1969] did not follow this convention, and in $V_N$ we write

$$\sigma_{i|j} = \sigma_{|i|j} \neq \sigma_{ij} .$$

We will denote covariant differentiation with respect to $\Gamma_{ij}^{\ \ k}$ by "," and with respect to $\hat{\Gamma}_{ij}^{\ \ k}$ by ";" . Directional derivatives will be denoted by "/" , e.g., the directional derivative of f in the direction $\underset{\sim}{\lambda}$ is

$$f_{/\lambda} := f_j \lambda^j .$$

The Beltrami differential parameter of the first kind will be denoted by $\Delta_1 \sigma$ where

$$\Delta_1 \sigma := g^{ij} \sigma_i \sigma_j .$$

The Beltrami differential parameter of the second kind (or Laplace-Beltrami operator) is denoted by $\Delta_2 \sigma$ and given by

$$\Delta_2 \sigma := g^{ij} \sigma_{i,j} . \tag{3.8}$$

The usual Euclidean expressions for $\Delta_1 \sigma$ and $\Delta_2 \sigma$ are

$$\Delta_1 \sigma = (\underset{\sim}{\nabla} \sigma) \cdot (\underset{\sim}{\nabla} \sigma) , \tag{3.9}$$

where "$\cdot$" denotes the usual Euclidean inner product and

$$\Delta_2 \sigma = \underset{\sim}{\nabla} \cdot (\underset{\sim}{\nabla} \sigma) = \nabla^2 \sigma . \tag{3.10}$$

We see how (3.9) and (3.10) are related to (3.7) and (3.8) by denoting the Cartesian metric tensor by

$$g_{ij} = \delta_{ij} , \tag{3.11}$$

so that

$$\Delta_1 \sigma = \delta^{ij} \sigma_i \sigma_j$$

and

$$\Delta_2\sigma = g^{ij}\sigma_{i,j} .$$

Note that in $V_N$

$$\sigma_{ij} \neq \sigma_{|i|j}$$

and

$$\sigma_{ij} \neq \sigma_{i,j} ;$$

note also that

$$\sigma_{|i|j} = \sigma_{i|j} ;$$

and by definition one writes

$$\sigma_{ij} := \sigma_{i,j} - \sigma_i\sigma_j .$$

The Riemann tensor of $V_N$ will be denoted by $R_{hijk}$, the Ricci tensor by $R_{ij}$, and the scalar curvature by $R$. The Riemann tensor of $\hat{V}_N$ is given by

$$\hat{R}_{hijk} = e^{2\sigma}(R_{hijk} + \Sigma_{hijk}) \tag{3.12}$$

where

$$\Sigma_{hijk} := g_{hk}\sigma_{ij} + g_{ij}\sigma_{hk} - g_{hj}\sigma_{ik} - g_{ik}\sigma_{hj} - \Delta_1\sigma g_{hijk} , \tag{3.13}$$

and

(3.14) $$g_{hijk} := g_{hj}g_{ik} - g_{hk}g_{ij} \ .$$

Some useful contractions of $\Sigma_{hijk}$ and $g_{hijk}$ are given by

(3.15) $$\Sigma_{ij} := \Sigma^{h}_{ijh} = (N - 2)\sigma_{ij} + (\Delta_2\sigma + (h - 2)\Delta_1\sigma)g_{ij} \ ,$$

or alternately

(3.16) $$\Sigma_{ij} = (N - 2)[\sigma_{ij} + g_{ij}\Delta_1\sigma] + \frac{1}{2(N-1)} \Sigma g_{ij} \ ,$$

where

(3.17) $$\Sigma := g^{ij}\Sigma_{ij} = \Sigma^{i}_{i} = 2(N - 1)\Delta_2\sigma + (N - 1)(N - 2)\Delta_1\sigma \ ,$$

(3.18) $$g^{hk}g_{hijk} = (1 - N)g_{ij} \ ,$$

(3.19) $$g^{ij}g^{hk}g_{hijk} = (1 - N)N \ ,$$

(3.20) $$g^{h}{}_{ijk} = \delta^{h}_{j}g_{ik} - \delta^{h}_{k}g_{ij} =$$

$$\begin{vmatrix} \delta^{h}_{j} & \delta^{h}_{k} \\ g_{ij} & g_{ik} \end{vmatrix} \ ,$$

(3.21) $$g_{hijk} = \delta^{pq}_{hi}g_{pj}g_{qk} = \begin{vmatrix} g_{hj} & g_{hk} \\ g_{ij} & g_{ik} \end{vmatrix} \ ,$$

and

(3.22) $$g^{hi}{}_{jk} \equiv \delta^{hi}_{jk} := \delta^{h}_{j}\delta^{i}_{k} - \delta^{h}_{k}\delta^{i}_{j} \ ,$$

and where $\delta^i_j$ is the Kronecker delta.

The relations (3.12)-(3.22) make it possible to calculate the Ricci tensor and scalar curvature of $\hat{V}_N$. The Ricci tensor of $\hat{V}_N$, $\hat{R}_{ij}$, is given by

$$\text{(3.23)} \qquad \hat{R}_{ij} = R_{ij} + (N-2)\sigma_{ij} + g_{ij}(\Delta_2\sigma + (N-2)\Delta_1\sigma) ,$$

and $\hat{R}$, the scalar curvatures of $\hat{V}_N$, is given by

$$\text{(3.24)} \qquad \hat{R} = e^{-2\sigma}(R + 2(N-1)\Delta_2\sigma + (N-1)(N-2)\Delta_1\sigma) .$$

As usual

$$\text{(3.25)} \qquad R_{ij} := R^h{}_{ijh} = g^{hk}R_{hijk} ,$$

and

$$\text{(3.26)} \qquad R := g^{ij}R_{ij} = R^i{}_i .$$

We also have

$$\text{(3.27)} \qquad \hat{g}_{ij}\hat{R} = g_{ij}(R + 2(N-1)\Delta_2\sigma + (N-1)(N-2)\Delta_1\sigma) ,$$

from which it follows that

$$\text{(3.28)} \qquad \hat{g}_{ij}\hat{R} - g_{ij}R - (N-1)(N-2)\Delta_1\sigma g_{ij} = 2(N-1)\Delta_2\sigma g_{ij} ,$$

viz.

$$(3.29) \qquad \Delta_2\sigma g_{ij} = \frac{1}{2(N-1)}(\hat{g}_{ij}\hat{R} - g_{ij}R) - \frac{(N-2)}{2}\Delta_1\sigma g_{ij} .$$

Thus we have

$$(3.30) \qquad \begin{aligned} \hat{R}_{ij} - R_{ij} &= (N-2)\sigma_{ij} + ((N-1)\Delta_1\sigma + \Delta_2\sigma)g_{ij} \\ &= (N-2)\sigma_{ij} + \frac{1}{2(N-1)}(\hat{g}_{ij}\hat{R} - g_{ij}R) \\ &\quad - \frac{(N-2)}{2}\Delta_1 g_{ij} , \end{aligned}$$

and therefore

$$(3.31) \qquad \begin{aligned} (N-2)\sigma_{ij} &= \hat{R}_{ij} - R_{ij} - \frac{1}{2(N-1)}(\hat{g}_{ij}\hat{R} - g_{ij}R) \\ &\quad - \frac{(N-2)}{2}\Delta_1\sigma g_{ij} . \end{aligned}$$

Upon replacing $\sigma_{ij}$ in [illegible]$_{hijk}$ via (3.31) we obtain

$$(3.32) \qquad \hat{C}^h{}_{ijk} = C^h{}_{ijk} .$$

This shows that $C^h{}_{ijk}$, defined by

$$(3.33) \qquad \begin{aligned} C^h{}_{ijk} &:= R^h{}_{ijk} + \frac{1}{(N-2)}(\delta^h_j R_{ik} - \delta^h_k R_{ij} + g_{ik}R^h{}_j - g_{ij}R^h_k) \\ &\quad - \frac{R}{(N-1)(N-2)}(\delta^h_j g_{ik} - \delta^h_k g_{ij}) , \end{aligned}$$

is invariant under a conformal map. It should be noted that

$$(3.34) \qquad C_{hijk} := g_{hp}C^{p}{}_{ijk} =$$

$$R_{hijk} + \frac{1}{(N-2)}(g_{hj}R_{ik} - g_{hk}R_{ij} + g_{ik}R_{hj} - g_{ij}R_{hk})$$

$$- \frac{R}{(N-1)(N-2)} g_{hijk}$$

is not a conformal invariant since

$$(3.35) \qquad \hat{C}_{hijk} = e^{2\sigma}C_{hijk} .$$

The tensor $C^{h}{}_{ijk}$ is called the Weyl conformal curvature tensor. The Weyl tensor is completely traceless, i.e., all contractions vanish. It satisfies the usual symmetries:

$$(3.36) \qquad C^{h}{}_{ijk} = -C^{h}{}_{ikj}$$

$$(3.37) \qquad C_{hijk} = -C_{ihjk} = C_{jkhi}$$

and the <u>algebraic</u> Bianchi identity,

$$(3.38) \qquad C^{h}{}_{ijk} + C^{h}{}_{kij} + C^{h}{}_{jki} = 0 .$$

However, it does <u>not</u> satisfy the <u>differential</u> Bianchi identity. To see this we introduce the Finzi tensor

$$(3.39) \qquad L_{ijk} := R_{ij,k} - R_{ik,j} + 2(\frac{1}{(N-1)})(g_{ik}R_{,j} - g_{ij}R_{,k}) ,$$

which satisfies

$$(3.40) \qquad L_{ijk} = -L_{ikj} ,$$

and

$$(3.41) \qquad L^h{}_{jk} := g^{hi} L_{ijk} .$$

Then the conformal analog of the differential Bianchi identity is

$$(3.42) \qquad C^h{}_{ijk,\ell} + C^h{}_{i\ell j,k} + C^h{}_{ik\ell,j} =$$

$$\frac{1}{(N-2)} (\delta^h_j L_{ik\ell} + \delta^h_k L_{i\ell j} + \delta^h_\ell L_{ijk} + g_{ik} L^h{}_{j\ell} + g_{i\ell} L^h_{kj} + g_{ij} L^h{}_{\ell k}) .$$

The singly contracted Bianchi identity is

$$(3.43) \qquad R_{ij,k} - R_{ik,j} + g^{hm} R_{mikj,h} = 0$$

and the doubly contracted Bianchi identity is

$$(3.44) \qquad R^i{}_{k,i} = \frac{1}{2} R_{,k} .$$

From this it follows that

$$(3.45) \qquad L^i{}_{ik} = 0 .$$

Moreover, an interesting contraction of (3.42) is obtained by using (3.45)

$$(3.46) \qquad C^h{}_{ijk,h} = \frac{N-3}{N-2} L_{ijk} .$$

We now introduce two additional tensors -- the Einstein tensor and the Cotton tensor -- which are important tensors in arbitrary dimension but have particular significance in dimensions 2, 3, and 4.

The Einstein tensor is defined by

$$S_{ij} := R_{ij} - \frac{1}{2} g_{ij}R \,, \tag{3.47}$$

and the Cotton tensor by

$$\Lambda_{ij} := R_{ij} - \frac{1}{4} g_{ij}R \,. \tag{3.48}$$

## §4. SPECIAL VALUES OF "N"

We first consider the case $N = 2$.

The elimination procedure used in §3 to obtain a conformal curvature tensor breaks down for $\hat{R}_{ij}$, $R_{ij}$, $\hat{R}$, and $\hat{g}_{ij}\hat{R}$ when $N = 2$. We first show that every $V_2$ is an Einstein space (recall that an Einstein space is a Riemannian space such that

$$R_{ij} = \frac{R}{N} g_{ij} \,, \tag{4.1}$$

or in the case $N = 2$

$$R_{ij} - \frac{1}{2} g_{ij}R = S_{ij} = 0) \,. \tag{4.2}$$

In a $V_2$ the curvature tensor has the canonical form

$$R^h{}_{ijk} = K(\delta^h_j g_{ik} - g_{ij}\delta^h_k) \tag{4.3}$$

and so

$$R_{ij} = K(g_{ij} - 2g_{ij}) = -Kg_{ij} \,. \tag{4.4}$$

Hence we have

$$(4.5) \qquad R = -2K ,$$

or

$$(4.6) \qquad K = -\frac{1}{2} R .$$

Therefore,

$$(4.7) \qquad R_{ij} = \frac{R}{2} g_{ij}$$

so that every $V_2$ is an Einstein space. Now observe that when $N = 2$ we have

$$(4.8) \qquad \hat{R}_{ij} = R_{ij} + \Lambda_2 \sigma g_{ij} ,$$

$$(4.9) \qquad \hat{R} = e^{-2\sigma}[R + 2\Lambda_2 \sigma] ,$$

$$(4.10) \qquad \hat{g}_{ij} = g_{ij}[R + 2\Lambda_2 \sigma] ,$$

so that we have

$$(4.11) \qquad \hat{R}_{ij} - R_{ij} - \frac{1}{2}(\hat{g}_{ij}\hat{R} - g_{ij}R) = 0 ,$$

or using the definition of $S_{ij}$ ,

$$(4.12) \qquad \hat{S}_{ij} - S_{ij} = 0 .$$

But since $V_2$ is an Einstein space, the elimination scheme leading to a conformal curvature tensor fails at this stage. This is why $C^h{}_{ijk}$ and $C_{hijk}$ in (3.33) and (3.34) are undefined when $N = 2$ .

On the other hand is well known that **every** $V_2$ is locally conformal to **every** other $V_2$ .

Now we discuss $N = 3$ .

The dimension $N = 3$ is much more complicated than the dimension $N = 2$ . Nevertheless, the elimination procedure fails to lead to a conformal curvature tensor and in fact we show that the conformal curvature tensor discussed in §3 vanishes identically in a $V_3$ .

Before we show that the elimination method fails when $N = 3$ , we derive several canonical forms for the curvature tensor in a $V_3$ . The simplest derivation of a canonical form for $R_{hijk}$ in a $V_3$ is due to WILKES/ZUND[1978] who complete a problem in McCONNELL[1931]. To do this we employ the Levi-Civita dualizors $\epsilon^{ijk}$ and contract on the skew symmetric pairs of indices in the curvature tensor $R_{hijk}$ , viz.

$$\epsilon^{ipq}\epsilon^{jrs}R_{pqrs} \tag{4.13}$$

to create a second-order tensor. $\epsilon^{ipq}\epsilon^{jrs}$ is expressible as a $3 \times 3$ determinant in the metric tensor $g_{ij}$

$$\epsilon^{ipq}\epsilon^{jrs} \equiv \begin{vmatrix} g^{ij} & g^{ir} & g^{is} \\ g^{pj} & g^{pr} & g^{ps} \\ g^{qj} & g^{qr} & g^{qs} \end{vmatrix} . \tag{4.14}$$

Upon multiplication of (4.13) by the numerical factor of $\frac{1}{4}$ and using (4.14) we obtain

$$S^{ij} = \frac{1}{4}\epsilon^{ipq}\epsilon^{jrs}R_{pqrs} , \tag{4.15}$$

where $S^{ij}$ is the contravariant Einstein tensor. McCONNELL[1931] calls this tensor the Lamé tensor. However, Lamé's work was exclusively three-dimensional and since this tensor has applications in dimensions $N \neq 3$, the name "Einstein tensor" seems more appropriate. Equation (4.15) has an "inverse", namely

$$R_{hijk} = \epsilon_{hip}\epsilon_{jkq}S^{pq} . \tag{4.16}$$

Equations (4.15)-(4.16) enable us to write down several <u>**canonical forms**</u> for $R_{hijk}$. First,

$$\begin{aligned} R_{hijk} = {} & g_{hk}S_{ij} + g_{ij}S_{hk} - g_{hj}S_{ik} \\ & - g_{ik}S_{hj} - \frac{R}{2}\, g_{hijk} \end{aligned} \tag{4.17}$$

which we call the <u>**S-representation**</u> of $R_{hijk}$. Second, from the definition of the Cotton tensor and (4.18) we have

$$R_{hijk} = g_{hk}\Lambda_{ij} + g_{ij}\Lambda_{hk} - g_{hj}\Lambda_{ik} - g_{ik}\Lambda_{hj} , \tag{4.18}$$

which we call the <u>**Λ-representation**</u> of $R_{hijk}$. Third, upon expansion of (4.18), or (4.19), we obtain

$$R_{hijk} = g_{hk}R_{ij} + g_{ij}R_{hk} - g_{hj}R_{ik} - g_{ik}R_{hj} - \frac{R}{2}\, g_{hijk} , \tag{4.19}$$

which is the most common canonical form found in the literature. It shows that the Weyl tensor, $C_{hijk}$, vanishes identically in a $V_3$. However, the S-representation and the Λ-representation are more useful, and the $\Lambda$-representation is particularly nice as it does not involve a factor of $g_{hijk}$. Contraction of (4.19) via $g^{hk}$ does not yield an expression for $R_{ij}$ ; however, contraction of (4.17) gives

$$R_{ij} = S_{ij} + g_{ij}S + Rg_{ij} , \tag{4.20}$$

so that

$$S = \frac{-R}{2} . \tag{4.21}$$

Hence we obtain the Einstein tensor of §3! Likewise contraction of (4.18) yields

$$R_{ij} = \Lambda_{ij} + g_{ij}\Lambda , \tag{4.22}$$

so that

$$\Lambda = \frac{R}{4} , \tag{4.23}$$

and we obtain the Cotton tensor!

The crucial step in the elimination scheme in §3 was the solvability of (3.30) for $\sigma_{ij}$ . When $N = 3$ , (3.31) becomes

$$\sigma_{ij} = \dot{R}_{ij} - R_{ij} - \frac{1}{4} (\dot{g}_{ij}\dot{R} - g_{ij}R) - \frac{1}{2}\Lambda_1\sigma g_{ij} \tag{4.24}$$

and using the Cotton tensor (3.48) we have

$$\sigma_{ij} = \dot{\Lambda}_{ij} - \Lambda_{ij} - \frac{1}{2}\Lambda_1\sigma g_{ij} . \tag{4.25}$$

Substituting this expression in $C_{hijk}$ then gives

(4.26) $$\hat{R}_{hijk} = e^{2\sigma}(R_{hijk} + \Sigma_{hijk})$$

$$= e^{2\sigma}R_{hijk} + e^{2\sigma}[g_{hk}(\hat{\Lambda}_{ij} - \Lambda_{ij}$$

$$- \frac{1}{2}\Lambda_1{}^\sigma g_{ij}) - g_{hj}(\hat{\Lambda}_{ik} - \Lambda_{ik}$$

$$- \frac{1}{2}g_{ik}\Lambda_1{}^\sigma) - g_{ik}(\hat{\Lambda}_{hj} - \Lambda_{hj}$$

$$- \frac{1}{2}g_{hj}\Lambda_1{}^\sigma] - e^{2\sigma}\Lambda_1{}^\sigma g_{hijk} .$$

Therefore,

(4.27) $$\hat{R}_{hijk} - \hat{g}_{hj}\hat{\Lambda}_{ij} - \hat{g}_{ij}\hat{\Lambda}_{hk} + \hat{g}_{hj}\hat{\Lambda}_{ik} + \hat{g}_{ik}\hat{\Lambda}_{hj} =$$

$$e^{2\sigma}(R_{hijk} - g_{hk}\Lambda_{ij} - g_{ij}\Lambda_{hk} + g_{hj}\Lambda_{ik} + g_{ik}\Lambda_{hj})$$

but this reduces to $0 = 0$ by the $\Lambda$-representation of $R_{hijk}$ (4.18). Thus the elimination scheme again fails to give a conformal curvature tensor in 3 dimensions.

The foregoing shows not only that the Weyl tensor vanishes identically in a $V_3$, but also that the Cotton tensor and the Einstein tensor naturally occur. The former plays an important role when we consider integrability conditions in conformally flat spaces in the next section.

## §5. INTEGRABILITY CONDITIONS AND CONFORMALLY FLAT SPACES

**Definition 5.1**. If $V_N$ and $\hat{V}_N$ are conformally related and $\hat{V}_N$ is flat, i.e., $\hat{V}_N$ is locally isometric to N-dimensional Euclidean space, $E_N$, then $V_N$ is conformally *flat*. A conformally flat space will be denoted by $C_N$ .

We seek to develop necessary and sufficient conditions for a $V_N$ to be a $C_N$ . This problem will be addresed in two parts, first, $N > 3$ , and, second, $N = 3$ . The case $N = 2$ is not of particular interest since as noted previously any $V_2$ is conformal to any other $V_2$ .

Preliminary to this problem we suppose $N > 2$ and assume that the curvature tensor of $V_N$ can be represented in the form

$$R_{hijk} = g_{hk}X_{ij} + g_{ij}X_{hk} - g_{hj}X_{ik} - g_{ik}X_{hj} \tag{5.1}$$

where

$$X_{ij} = X_{ji} , \tag{5.2}$$

and

$$X := g^{ij}X_{ij} \tag{5.3}$$

are to be determined. The representation (5.1) is called an <u>**X-representation**</u> of $R_{hijk}$ . From (5.1) we have

$$R_{ij} := g^{hk}R_{hijk} = NX_{ij} + g_{ij}X - X_{ij} - X_{ij} , \tag{5.4}$$

so that

$$R_{ij} = (N - 2)X_{ij} + Xg_{ij} , \tag{5.5}$$

and

$$R := g^{ij}R_{ij} = 2(N - 1)X . \tag{5.6}$$

Hence, we obtain

$$(5.7) \qquad X_{ij} = \frac{1}{(N-2)} (R_{ij} - g_{ij}X) ,$$

or

$$(5.8) \qquad X_{ij} = \frac{1}{N-2} (R_{ij} - \frac{R}{2(N-1)} g_{ij}) .$$

We may draw the following conclusions:

(i) If $R_{hijk}$ has an X-representation, then it is solvable for the X-tensor and its contractions.

(ii) Expanding the X-representation yields

$$(5.9) \qquad R_{hijk} = \frac{1}{(N-2)} (g_{hk}R_{ij} + g_{ij}R_{hk} - g_{hj}R_{ik} - g_{ik}R_{hj} - \frac{R}{(N-1)} g_{hijk}) .$$

We have thus proved

**Theorem 5.2.** For $N \geq 3$ the Weyl tensor vanishes if and only if the curvature tensor has an X-representation.

Let $V_N$ and $\hat{V}_N$ be conformally related. From (3.12) and (3.13) we have

$$(5.10) \qquad \hat{R}_{hijk} = e^{2\sigma}(R_{hijk} + g_{hk}\sigma_{ij} + g_{ij}\sigma_{hk} - g_{hj}\sigma_{ik} - g_{ik}\sigma_{hj} - g_{hijk}\Lambda_1\sigma) ,$$

and from (3.23) and (3.24) we have

$$(5.11) \qquad \hat{R}_{ij} = R_{ij} + (N-2)\sigma_{ij} + g_{ij}(\Delta_2\sigma + (N-2)\Delta_1\sigma)$$

and

$$(5.12) \qquad \hat{R} = e^{-2\sigma}(R + 2(N-1)\Delta_2\sigma + (N-1)(N-2)\Delta_1\sigma) \ .$$

Hence we find that

$$(5.13) \qquad \hat{g}_{ij}\hat{R} = g_{ij}(R + 2(N-1)\Delta_2\sigma + (N-1)(N-2)\Delta_1\sigma).$$

Eliminating $\Delta_2\sigma$ from (5.11) and (5.13) we obtain

$$(5.14) \qquad \sigma_{ij} = \frac{1}{(N-2)}\left[(\hat{R}_{ij} - R_{ij}) - \frac{1}{2(N-1)}(\hat{g}_{ij}\hat{R} - g_{ij}R)\right] - \frac{1}{2} g_{ij}\Delta_1\sigma = \hat{X}_{ij} - X_{ij} - \frac{1}{2} g_{ij}\Delta_1\sigma \ .$$

Upon replacing this value of $\sigma_{ij}$ in (5.10) we see that $R_{hijk}$ has an X-representation if and only if $\hat{R}_{hijk}$ has an $\hat{X}$-representation.

Now suppose $N > 3$ and let $\hat{R}_{hijk} = 0$ . Then $\hat{R}_{ij} = 0$ and $\hat{R} = 0$ , so that

$$(5.15) \qquad \sigma_{ij} = -X_{ij} - \frac{1}{2} g_{ij}\Delta_1\sigma \ ,$$

or

$$(5.16) \qquad \sigma_{i,j} = \sigma_i\sigma_j - X_{ij} - \frac{1}{2} g_{ij}\Delta_1\sigma \ .$$

Replacement of this in (5.10) shows that $C_{hijk} = 0$ and that $R_{hijk}$ has an X-representation. That is, $C_{hijk} = 0$ in a $C_N$.

Conversely, suppose that $C_{hijk} = 0$ and that $\sigma$ is defined by the differential equation (5.16). For $\sigma$ to exist $\sigma_{i,j}$ must be symmetric in $i$ and $j$ and $\sigma_i$ must satisfy the following <u>integrability conditions</u>:

$$(5.17) \qquad \sigma_{i,j,k} - \sigma_{i,k,j} = \sigma_h R^h{}_{ijk} \, .$$

It is easily seen that $\sigma_{i,j}$ is symmetric, and differentiating (5.16) we obtain

$$(5.18) \quad \sigma_{i,j,k} = \sigma_{i,k}\sigma_j + \sigma_i\sigma_{j,k} - X_{ij,k} - \frac{1}{2} g_{ij}(\Lambda_1\sigma)_{,k} \, .$$

Interchanging $j$ and $k$ gives

$$(5.19) \quad \sigma_{i,k,j} = \sigma_{i,j}\sigma_k + \sigma_i\sigma_{j,k} - X_{ik,j} - \frac{1}{2} g_{ik}(\Lambda_1\sigma)_{,j} \, .$$

Thus, we have

$$(5.20) \qquad \sigma_{i,j,k} - \sigma_{i,k,j} =$$

$$\sigma_{i,k}\sigma_j - \sigma_{i,j}\sigma_k + X_{ik,j} - X_{ij,k} + \frac{1}{2}(g_{ik}(\Lambda_1\sigma)_{,j} - g_{ij}(\Lambda_1\sigma)_{,k})$$

and using (5.16) and Theorem 5.2, the integrability conditions become merely

$$(5.21) \qquad X_{ij,k} - X_{ik,j} = 0 \, .$$

But these follow from (3.46) when $C_{hijk} = 0$. Thus, a function $\sigma$

satisfying (5.16) exists. If we define $\hat{V}_N$ by $\hat{g}_{ij} = e^{2\sigma} g_{ij}$ where $\sigma_j$ is the function of (5.16), then a straightforward computation shows that $\hat{R}_{hijk} = 0$. Thus, we have the following theorem.

**Theorem 5.3** (Schouten's Lemma) SCHOUTEN[1921]. A $V_N(N > 3)$ is a $C_N$ if and only if the curvature tensor of $V_N$ has an X-representation.

We also have

**Corollary 5.4**. A $V_N(N > 3)$ is a $C_N$ if and only if the Weyl conformal curvature tensor vanishes.

In §3 we defined the Finzi tensor:

$$L_{ijk} = R_{ij,k} - R_{ik,j} + \frac{1}{2(N-1)} (g_{ik}R_{,j} - g_{ij}R_{,k}) \tag{5.22}$$

and we now observe that

$$L_{ijk} = (N - 2)(X_{ij,k} - X_{ik,j}) . \tag{5.23}$$

If $V_N$ and $\hat{V}_N$ are conformally related, then the Finzi tensor in $\hat{V}_N$ is

$$\hat{L}_{ijk} = (N - 2)(\hat{X}_{ij;k} - \hat{X}_{ik;j}) . \tag{5.24}$$

Writing (5.24) in terms of un-hatted objects we have

$$(5.25) \quad \hat{L}_{ijk} = (N-2)(\hat{X}_{ij;k} - \hat{X}_{ik,j})$$

$$= L_{ijk} + (N-2)[\sigma_j X_{ik} - \sigma_k X_{ij} + g_{ik} g^{mp} \sigma_p X_{jm}$$

$$- g_{ij} g^{mp} \sigma_p + [\sigma_{i,j,k} - \sigma_{i,k,j} - \sigma_k \sigma_{ij}$$

$$+ \sigma_j \sigma_{ik}] + (g_{ik} g^{mp} \sigma_p \sigma_{jm} -$$

$$g_{ij} g^{mp} \sigma_p \sigma_{km}) + \Lambda_1 \sigma (g_{ik} \sigma_j - g_{ij} \sigma_k)$$

$$+ \frac{1}{2} g_{ij} (\Lambda_1 \sigma)_{,k} - \frac{1}{2} g_{ik} (\Lambda_1 \sigma)_{,j}] \, ,$$

which reduces to

$$(5.26) \quad \hat{L}_{ijk} = L_{ijk} + \sigma_j X_{ik} - \sigma_k X_{ij} + g_{ik} g^{mp} \sigma_p X_{jm}$$

$$- g_{ij} g^{mp} \sigma_p X_{km} + \sigma_{i,j,k} - \sigma_{i,k,j} \, .$$

By the integrability conditions (5.17) on $\sigma$ we obtain

$$(5.27) \quad \hat{L}_{ijk} = L_{ijk} + \sigma_i X_{ik} - \sigma_k X_{ij} + g_{ik} g^{mp} \sigma_p X_{jm}$$

$$- g_{ij} g^{mp} \sigma_p X_{km} + \sigma_h R^h{}_{ijk} \, .$$

Hence, if $R_{hijk}$ has an X-representation, then $\hat{L}_{ijk} = L_{ijk}$ ; and if $\hat{L}_{ijk} = L_{ijk}$ , then $R_{hijk}$ has an X-representation. Thus, we have proven

**Theorem 5.5** (Finzi's Theorem) FINZI[1922]. In $V_N (N \geq 3)$ the Finzi tensor is **conformal invariant** if and only if the curvature tensor has an X-representation.

**Corollary 5.6.** In $V_N (N > 3)$ the Finzi tensor is a zero tensor if and only if $V_N$ is a $C_N$ .

We now consider the case $N = 3$ .

By Theorem 5.5 we know that the Finzi tensor is a conformal invariant in $V_3$ since the curvature tensor of $V_3$ always has an X-representation, viz $X_{ij} = \Lambda_{ij}$ , the Cotton tensor. If we define $\sigma$ by (5.16), then the integrability conditions are

$$L_{ijk} = 0 \ . \tag{5.28}$$

If we insert (5.15) into (5.10) we have

$$\hat{R}_{hijk} = 0 \ . \tag{5.29}$$

If $\hat{R}_{hijk} = 0$ , we have $\hat{R}_{ij} = 0$ and $\hat{R} = 0$ and so $\hat{X}_{ij} = 0$ . Therefore, we find that

$$\hat{L}_{ijk} = 0 \ . \tag{5.30}$$

Hence, we have

$$L_{ijk} = 0 \tag{5.31}$$

since $L_{ijk}$ is conformal invariant. Thus we obtain

**Theorem 5.6.** A $V_3$ is conformally flat if and only if the Finzi tensor is a zero tensor.

NB: There is <u>no</u> Weyl tensor in a $V_3$ .

To summarize, we have obtained necessary and sufficient conditions for a $V_N (N \geq 3)$ to be conformally flat. Additionally, we have derived an often-neglected result (Theorem 5.5) on the Finzi tensor.

## CHAPTER III

## TRIPLY ORTHOGONAL SYSTEMS

### §1. TRIPLY ORTHOGONAL SYSTEMS OF SURFACES

Since a point on a surface is determined by two parameters, a point in space may be determined by three parameters or <u>curvilinear coordinates</u>. In $E_3$, for example, we may transform our Cartesian $x^i = (x^1, x^2, x^3)$ by means of the equations

$$x^i = x^i(u,v,w) \qquad i = 1, 2, 3 \tag{1.1}$$

where $u^i = (u,v,w)$ are curvilinear coordinates. The surfaces $u = C_1$, $v = C_2$, $w = C_3$, where $C_1, C_2, C_3$ are constants, are the <u>coordinate surfaces</u> and they intersect in <u>coordinate lines</u>. Examples are cylindrical and spherical coordinates, i.e.,

$$x^1 = u \cos v \ , \ x^2 = u \sin v \ , \ x^3 = w \tag{1.2}$$

and

$$x^1 = u \cos v \cos w \ , \ x^2 = u \cos v \sin w, \ x^3 = u \sin v \ , \tag{1.3}$$

respectively. The line element in our space will then take the form

$$ds^2 = \delta_{ij} \frac{\partial x^i}{\partial u^m} \frac{\partial x^j}{\partial u^n} du^m du^n \ . \tag{1.4}$$

Of particular interest are those coordinate systems for which

$$\delta_{ij} \frac{\partial x^i}{\partial u^m} \frac{\partial x^j}{\partial u^n} = 0 \qquad n \neq m \ . \tag{1.5}$$

In this case the line element becomes

$$(1.6) \qquad ds^2 = \delta_{ij}\left[\frac{\partial x^i}{\partial u}\frac{\partial x^j}{\partial u}du^2 + \frac{\partial x^i}{\partial u}\frac{\partial x^j}{\partial v}dv^2 + \frac{\partial x^i}{\partial w}\frac{\partial x^j}{\partial w}dw^2\right],$$

and the quantities

$$\delta_{ij}\frac{\partial x^i}{\partial u^m}\frac{\partial x^j}{\partial u^m} \qquad \text{(NS on } m!)$$

are conventionally denoted by

$$(1.7) \qquad h_m{}^2 := \delta_{ij}\frac{\partial x^i}{\partial u^m}\frac{\partial x^j}{\partial u^m} \qquad \text{(NS on } m!)$$

so that

$$(1.8) \qquad ds^2 = \sum_m h_m{}^2 du^m$$

$$= h_1{}^2du^2 + h_2{}^2dv^2 + h_3{}^2dw^2$$

and the functions $h_m$ are called <u>scale factors</u>. The coordinate surfaces of an orthogonal coordinate system give rise to a triply orthogonal family of surfaces. The simplest example are Cartesian, cylindrical, and spherical coordinates where the coordinate surfaces are respectively planes; planes and cylinders; and planes, spheres, and cones. We give one less trivial example.

Example: Confocal quadrics are given by

$$(1.9) \qquad \frac{x^2}{a^2-u} + \frac{y^2}{b^2-u} + \frac{z^2}{c^2-u} = 1 \qquad a^2 < b^2 < c^2 .$$

When $u < a^2$ (1.9) gives ellipsoids, when $a^2 < u < b^2$ it represents hyperboloids of one sheet and when $b^2 < u < c^2$ it represents hyperboloids of two sheets. If we re-write (1.9) as

$$\frac{x^2}{a^2-u} + \frac{y^2}{b^2-u} + \frac{z^2}{c^2-u} = 1 \qquad u < a^2 < b^2 < c^2$$

$$\frac{x^2}{a^2-v} + \frac{y^2}{b^2-v} + \frac{z^2}{c^2-v} = 1 \qquad a^2 < v < b^2 < c^2 \tag{1.10}$$

$$\frac{x^2}{a^2-w} + \frac{y^2}{b^2-w} + \frac{z^2}{c^2-w} = 1 \qquad a^2 < b^2 < w < c^2 ,$$

the line element takes the form

$$ds^2 = h_1{}^2du^2 + h_2{}^2dv^2 + h_3{}^2dw^2 ,$$

where

$$h_1{}^2 = \frac{1}{4} \left( \frac{(u-v)(u-w)}{(a^2-u)(b^2-u)(c^2-u)} \right)$$

$$h_2{}^2 = \frac{1}{4} \left( \frac{(w-v)(u-v)}{(a^2-v)(b^2-v)(c^2-v)} \right) \tag{1.11}$$

$$h_3{}^2 = \frac{1}{4} \left( \frac{(u-w)(v-w)}{a^2-w)(b^2-w)(c^2-w)} \right) .$$

We now give a proof of one of the fundamental theorems on triply orthogonal systems.

**Theorem 1.1** (Dupin): The curves of intersection of a triply orthogonal system of surfaces are lines of curvature on each of the surfaces. See McCONNELL[1931], p. 216.

Proof. For clarity we present the proof in the form of four assertions.

Assertion (i): We have

$$\tau_g = h_{\alpha\beta}\lambda^\alpha\lambda^\beta = \tau + \frac{d\theta}{ds} \qquad \alpha,\beta = 1,2 \tag{1.12}$$

where $\tau_g$ is the geodesic torsion, $\tau$ is the torsion, $h_{\alpha\beta}$ is the Euler tensor, $\theta$ is the angle between the unit surface normal and the unit principal normal of the curve and $\lambda^\alpha$ are the surface contravariant components of the unit tangent vector to the curve.

For a proof of this assertion see McCONNELL[1931], p. 214.

Assertion (ii): If a curve is the intersection of two surfaces which cut at a constant angle, then the geodesic torsions of the curve on the two surfaces have the same value.

Proof: Since the unit surface normals along these curves make a constant angle the rate of change of the angles $\theta$ , $\bar{\theta}$ with respect to arclength the unit surface normals and the unit principal normal must be the same, hence

$$\tau_{g1} = \tau + \frac{d\theta}{ds} = \tau + \frac{d\bar{\theta}}{ds} = \tau_{g2} \tag{1.13}$$

where $\tau_{g1}$ and $\tau_{g2}$ are the geodesic torsions on the first and second surfaces, respectively.

Assertion (iii) (Joachimsthal's Theorem): If a curve is the intersection of two surfaces which cut at a constant angle, then if the curve is a line of curvature, viz $\tau_g = 0$, on one surface it is a line of curvature on the other. See McCONNELL[1931], p. 215.

Proof: Follows immediately from (ii).

Assertion (iv): If two curves on a surface cut at right angles, then the sum of their geodesic torsions is zero.

Proof: Let $\lambda^\alpha$ be the components of the unit tangent vector of the curve on the surface and let $\rho^\alpha = \epsilon^{\gamma\alpha}\lambda_\gamma$ be the tangent vector of the perpendicular curve. Then

$$
\begin{aligned}
(1.14) \qquad \tau_{g\lambda} &= h_{\alpha\beta}\lambda^\alpha\lambda^\beta = \epsilon^{\gamma\delta}a_{\alpha\gamma}b_{\beta\delta}\lambda^\alpha\lambda^\beta \\
&= \epsilon^{\gamma\delta}b_{\beta\delta}\lambda_\gamma\lambda^\beta = b_{\beta\delta}\rho^\beta\lambda^\delta \\
&= b_{\beta\delta}\rho^\beta\epsilon^{\delta\gamma}\rho_\gamma = \epsilon^{\delta\gamma}a_{\alpha\gamma}b_{\beta\delta}\rho^\alpha\rho^\beta \\
&= -\epsilon^{\gamma\delta}a_{\alpha\gamma}b_{\beta\delta}\rho^\alpha\rho_\beta = -\tau_{g\rho} \ .
\end{aligned}
$$

Combining assertions (iii) and (iv) we get Dupin's Theorem.

Dupin's Theorem has a generalization due to DARBOUX[1910].

**Theorem 1.2** (Generalized Dupin): A necessary and sufficient condition that a third family of surfaces can be associated orthogonally -- viz be a component of a triply orthogonal system -- to a given pair of orthogonal families is that every surface of the two given families intersect each other in a line of curvature.

We postpone the proof of this theorem until Chapter IV §5, but we note that Generalized Dupin implies Dupin and that Dupin furnishes the necessity of Generalized Dupin.

## §2. THE CAYLEY-DARBOUX EQUATION[1)]

Let $E_3$ denote a three-dimensional Euclidean manifold having the line element

$$ds^2 = g_{ij} du^i du^j \ , \tag{2.1}$$

where $g_{ij} = h_i h_j$ are the curvilinear scale factors and $u^i = (u,v,w)$ are local curvilinear coordinates which are functions of the Cartesian coordinates $x^i = (x^1, x^2, x^3)$. Following DARBOUX[1910] the equations $u = C_1$, $v = C_2$, $w = C_3$, where $C_1$, $C_2$, $C_3$ are constants, are said to define a **Lamé system of surfaces**, i.e., a triply orthogonal system $\Sigma$, in $E_3$ and each equation defines a **Lamé family** of $\Sigma$.

It will be convenient to work with the functions $u,v,w : E_3 \to \mathbb{R}$ and to denote partial differentiation with respect to the coordinates $x^i$ by subscripts, viz. $u_i$, $u_{ij} = u_{ji}$, etc. We will use Cartesian tensor notation: Latin subscripts range from 1 to 3 and repeated subscripts obey the Einstein summation convention. The Schouten bracketing convention is employed to denote symmetrization and skew-symmetrization on subscripts.

---

[1]A version of the following material will appear in Tensor, N.S. See ZUND/MOORE[1986].

The orthogonality conditions for the curvilinear coordinated surfaces can then be written

$$(2.2) \qquad g_{12} = u_i v_i = 0 \ , \ g_{23} = v_i w_i = 0 \ , \ g_{13} = u_i w_i = 0 \ .$$

It is important to observe that the conditions in (2.2) are symmetric under a cyclic permutation of the parameter functions $u$, $v$, and $w$. We extend this observation to the <u>**symmetry principle**</u>: for any partial differential equation obtained by differentiation of (2.2) with respect to $x^j$ there are <u>two</u> analogous equations which arise by a cyclic permutation of $u$, $v$, and $w$.

Suppose that the Lamé family is defined by the surfaces $u = C_1$. Choose <u>any</u> one of the equations (2.2) involving $u$, e.g., $u_i v_i = 0$, and differentiate it with respect to $x^j$:

$$(2.3) \qquad u_i v_{ij} + u_{ij} v_i = 0 \ .$$

We note that (2.3) allows us to replace the second partials of $v$ with second partials of $u$ and we call this the <u>**replacement property**</u>.

Contracting (2.3) with $w_j$ and employing the symmetry property, we obtain two analogous equations. Upon adding two of these and subtracting the third we have

$$(2.4) \qquad v_{(i}w_{j)}u_{ij} + u_{[i}w_{j]}v_{ij} + u_{[i}v_{j]}w_{ij} = 0 \ .$$

But since this expression is symmetric in the subscripts $i$ and $j$, it reduces to

$$(2.5) \qquad v_i w_j u_{ij} = 0 \ .$$

Now calculating the partial derivative of (2.5) with respect to $x^k$, contracting the result with $u_k$, and using the replacement property on the partial derivatives of $v$ and $w$ yields

(2.6) $$A_{ij}v_i w_j = 0 ,$$

where

(2.7) $$A_{ij} := u_s u_{ijs} - 2u_{is}u_{js} .$$

Equation (2.6) may be regarded as a linear equation in the six unknowns:

(2.8) $$X_{ij} := v_{(i}w_{j)}$$

and to emphasize this, it is convenient to re-write (2.6) in the form

(2.9) $$A_{ij}X_{ij} = 0 .$$

If the unknowns $X_{ij}$ can be eliminated, we will obviously obtain a third-order partial differential equation involving the parameter $u$. To effect this elimination by a determinant we require five additional equations involving the $X_{ij}$. It turns out that these are easy to get. First, by using (2.8), (2.6) can be rewritten as

(2.10) $$u_{ij}X_{ij} = 0 ,$$

and by using (2.8) we also have

(2.11) $$X_{ii} = 0 .$$

The remaining three equations are given by

$$u_i X_{ij} = 0 , \tag{2.12}$$

by virtue of the orthogonality conditions (2.2). Thus the desired elimination is given by the determinant

$$\begin{vmatrix} A_{11} & A_{22} & A_{33} & 2A_{23} & 2A_{31} & 2A_{12} \\ u_{11} & u_{22} & u_{33} & 2u_{23} & 2u_{31} & 2u_{12} \\ 1 & 1 & 1 & 0 & 0 & 0 \\ u_1 & 0 & 0 & 0 & u_3 & u_2 \\ 0 & u_2 & 0 & u_3 & 0 & u_1 \\ 0 & 0 & u_3 & u_2 & u_1 & 0 \end{vmatrix} = 0 \tag{2.13}$$

which is the Cayley-Darboux equation. Substituting (2.7) we may write (2.13) as a difference of determinants

$$0 = u_s \begin{vmatrix} u_{11s} & u_{22s} & u_{33s} & 2u_{23s} & 2u_{31s} & 2u_{12s} \\ u_{11} & u_{22} & u_{33} & 2u_{23} & 2u_{31} & 2u_{12} \\ 1 & 1 & 1 & 0 & 0 & 0 \\ u_1 & 0 & 0 & 0 & u_3 & u_2 \\ 0 & u_2 & 0 & u_3 & 0 & u_1 \\ 0 & 0 & u_3 & u_2 & u_1 & 0 \end{vmatrix} \tag{2.14}$$

$$-2 \begin{vmatrix} u_{1s}u_{1s} & u_{2s}u_{2s} & u_{3s}u_{3s} & 2u_{2s}u_{3s} & 2u_{3s}u_{1s} & 2u_{1s}u_{2s} \\ u_{11} & u_{22} & u_{33} & 2u_{23} & 2u_{31} & 2u_{12} \\ 1 & 1 & 1 & 0 & 0 & 0 \\ u_1 & 0 & 0 & 0 & u_3 & u_2 \\ 0 & u_2 & 0 & u_3 & 0 & u_1 \\ 0 & 0 & u_3 & u_2 & u_1 & 0 \end{vmatrix}$$

It should be noted that these equations show that the Cayley-Darboux equation is linear in the third partial derivatives, cubic in the second partial derivatives, and quartic in the first partial derivatives of the function $u$. An explicit form for the Cayley-Darboux equation can be exhibited by expansion of the determinants in (2.13) or (2.14) and the results may be written as

$$\begin{aligned}(2.15) \quad \sum_{\alpha,\beta,\gamma} \{ &(u_s u_{\alpha\alpha s} - 2u_{\alpha s}u_{\alpha s})[u_\beta{}^3 u_{\gamma\alpha} + u_\beta u_\gamma{}^2 u_{\gamma\alpha} \\ &+ u_\alpha u_\gamma{}^2 u_{\beta\gamma} - (u_\gamma{}^3 u_{\alpha\beta} + u_\beta{}^2 u_\gamma u_{\alpha\beta} + u_\alpha u_\beta{}^2 u_{\beta\gamma}) \\ &+ u_\alpha u_\beta u_\gamma (u_{\beta\beta} - u_{\gamma\gamma})] + (u_s u_{\alpha\beta s} - u_{\alpha s} u_{\beta s}) \\ &[(u_{\gamma\gamma} - u_{\beta\beta})u_\alpha{}^2 + (u_{\alpha\alpha} - u_{\gamma\gamma})u_\beta{}^2 \\ &+ (u_{\alpha\alpha} - u_{\beta\beta})u_\gamma{}^2 + 2((u_\alpha{}^2 + u_\gamma{}^2)u_\beta u_{\beta\alpha} \\ &- (u_\beta{}^2 + u_\gamma{}^2)u_\alpha u_{\gamma\alpha})]\} = 0\end{aligned}$$

where the summation sign denotes the sum of a cyclic permutation of the Greek indices $\alpha, \beta, \gamma$ over 1, 2, 3. Inspection of (2.15) shows that the Cayley-Darboux equation contains 324 terms. The most general example given in DARBOUX[1910] contains only four terms.

## §3. THE THEOREM OF LIOUVILLE

Using the tools we have developed in this Chapter and in Chapter II we give a concise proof of one of the most amazing theorems in geometry and analysis.

**Theorem 3.1** (Liouville). The only conformal maps of $E_N$ to $E_N (N \geq 3)$ are **similarily** transformations (isometries and homothetic maps) and transformations by **reciprocal radii** (inversions in a sphere).

This is in sharp contrast to the case $N = 2$ where there is a rich supply of conformal maps. We give a proof, due to BIANCHI[1910] using the Lamé equations, that is valid in three dimensions only. For a proof for general $N$ see DUBROVIN/FOMENKO/NOVIKOV[1984].

Proof: In a triply orthogonal system, $u^i = (u,v,w)$ in $E_3$ the line element has the form

$$ds^2 = h_1{}^2 du^2 + h_2{}^2 dv^2 + h_3{}^2 dw^2 \tag{3.1}$$

where

$$\begin{aligned} ds_u &= h_1 du \\ ds_v &= h_2 dv \\ ds_w &= h_3 dw \end{aligned} \tag{3.2}$$

are the respective arclengths along the coordinate lines.

The Lamé equations are $S^{ij} = 0$ and following McCONNELL[1931], p. 156, we have

$$h_{123} = \frac{1}{h_1} h_{23}h_{12} + \frac{1}{h_3} h_{32}h_{13}$$

$$h_{231} = \frac{1}{h_3} h_{31}h_{23} + \frac{1}{h_1} h_{13}h_{21} \tag{3.3}$$

$$h_{312} = \frac{1}{h_1} h_{12}h_{31} + \frac{1}{h_2} h_{21}h_{32}$$

where $h_{ab} := h_{a|b}$ and $h_{abc} := h_{a|b|c}$ . We also have

$$\left(\frac{1}{h_1} h_{21}\right)_1 + \left(\frac{1}{h_2} h_{12}\right)_2 + \frac{1}{h_3^{\,2}} h_{13}h_{23} = 0$$

$$\left(\frac{1}{h_2} h_{32}\right)_2 + \left(\frac{1}{h_3} h_{23}\right)_3 + \frac{1}{h_1^{\,2}} h_{21}h_{31} = 0 \tag{3.4}$$

$$\left(\frac{1}{h_3} h_{13}\right)_3 + \left(\frac{1}{h_1} h_{31}\right)_1 + \frac{1}{h_2^{\,2}} h_{32}h_{12} = 0 \quad .$$

The Lamé equations are necessary and sufficient that the quadratic form (3.1) be reducible to

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \tag{3.5}$$

where $x^i = (x^1,x^2,x^3)$ .

Let $x^i = (x^1,x^2,x^3)$ be a local coordinate system in $E_3$ , with $\hat{x}^i = (\hat{x}^1,\hat{x}^2,\hat{x}^3)$ the coordinates of the image of $x^i$ under a conformal map, viz.

$$\hat{x}^i = f^i(x^1,x^2,x^3) \ . \tag{3.6}$$

Then the conformality of the mapping $E_3 \longrightarrow E_3$ requires that the ratio

(3.7) $$(\delta_{ij}d\hat{x}^i d\hat{x}^j)/(\delta_{k\ell}dx^k dx^\ell)$$

be independent of $dx^i$ and $d\hat{x}^i$, viz.

(3.8) $$\delta_{ij}d\hat{x}^i d\hat{x}^j = (1/\lambda^2)\delta_{ij}dx^i dx^j$$

where $\lambda$ is a function of $x^i$, $\lambda = \lambda(x^1,x^2,x^3)$.

In the Lamé equations (3.3) and (3.4), putting $x^i = u^i$ and each $h_i = 1/\lambda$, we obtain

(3.9) $$\lambda_{23} = 0\,,\quad \lambda_{31} = 0\,,\quad \lambda_{12} = 0\,,$$

where $\lambda_{ab} := \lambda_{|a|b}$, and

(3.10) $$\lambda_{11} + \lambda_{22} = \lambda_{22} + \lambda_{33} = \lambda_{33} + \lambda_{11} = \frac{1}{\lambda}\Lambda_1\lambda$$

where $\Lambda_1$ is the first Beltrami differential parameter with respect to $g_{ij} = \delta_{ij}$.

Equation (3.9) implies

(3.11) $$\lambda = X(x^1) + Y(x^2) + Z(x^3)$$

and substitution into (3.10) gives

(3.12) $$\ddot{X} = \ddot{Y} = \ddot{Z} = \frac{\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2}{2(X + Y + Z)} = k = \text{constant}$$

where "$\cdot$" denotes differentiation with respect to the independent variable.

Two cases occur according to whether

(i) $k = 0$ or (ii) $k \neq 0$ .

In (i), $k = 0$ implies $\dot{X} = \dot{Y} = \dot{Z} = 0$ hence $X$, $Y$, and $Z$ are constant, and therefore

$$\lambda = \text{const.} \tag{3.13}$$

This is the case of a similarity transformation, and $f : E_3 \longrightarrow E_3$ is an isometry or homothety.

In (ii) it is convenient to write $k = 2/c$ where $c \neq 0$ is a constant. Then integration of (3.12) yields

$$\begin{aligned} X &= \frac{1}{c}\left((x^1 - a^1)^2 + b^1\right) \\ Y &= \frac{1}{c}\left((x^2 - a^2)^2 + b^2\right) \\ Z &= \frac{1}{c}\left((x^3 - a^3)^2 + b^3\right) . \end{aligned} \tag{3.14}$$

But (3.12) also requires

$$(x^1 - a^1)^2 + (x^2 - a^2)^2 + (x^3 - a^3)^2 + b^1 + b^2 + b^3 = \sum_{i=1}^{3} (x^i - a^i)^2 , \tag{3.15}$$

viz.

$$\sum_{i=1}^{3} b^i = 0 . \tag{3.16}$$

Thus,

$$\lambda = \frac{2r}{c} , \tag{3.17}$$

where

$$r^2 = \sum_{i=1}^{3} (x^i)^2 . \tag{3.18}$$

Hence,

$$\delta_{ij} d\hat{x}^i d\hat{x}^j = \frac{c^2}{r^2} \delta_{ij} dx^i dx^j \tag{3.19}$$

so

$$\hat{x}^i = \frac{cx^i}{r} \tag{3.20}$$

which are the formulas for transformation by reciprocal radii, i.e., inversions. Hence the theorem is proved.

CHAPTER IV

DIFFERENTIAL GEODESY

## §1. INTRODUCTION

Martin Hotine in HOTINE[1966b] and in HOTINE[1969] sets forth the hypothesis that any sufficiently smooth function $\phi : E_3 \to \mathbb{P}$ with non-vanishing gradient can be a member of a triply orthogonal coordinate system for $E_3$. Hotine notes that this is at variance with classical differential geometry, which states that for a function to be a member of a triply orthogonal system it must satisfy the Cayley-Darboux equation of Chapter III, §2. Hotine then asserts that this equation is an identity when the Lamé equations are satisfied. See Chapter III, §3. Although it is difficult to determine when the Cayley-Darboux equation is satisfied we shall develop machinery, in particular, Ricci rotation coefficients, that will help in this question.

Hotine wants his assertion to be true since in geodesy the primary object of study is the shape of the geoid. If its potential function were a member of a triply orthogonal system, it would automatically provide a natural physical coordinatization of $E^3$ for the geoid. Unfortunately, as we will show in this Chapter, Hotine's argument is false and his hope for such a result cannot be realized.

## §2. CONFORMAL MAPPING AND ISOMETRIC IMMERSION

Throughout this Chapter, Greek indices range from 1 to 2 and Roman indices from 1 to 3.

Let $V_3$ be a 3-dimensional Riemannian space with metric tensor,

$g_{ij}$ , curvature tensor $R_{hijk}$ , Ricci tensor $R_{ij}$ , and Christoffel symbols $\Gamma^k_{ij}$ . Let $V_3$ be conformally related to $\hat{V}_3$ with metric tensor

$$(2.1) \qquad \hat{g}_{ij} = e^{2\sigma} g_{ij} ,$$

where $\sigma : V_3 \to \mathbb{R}$ is a smooth function. Let $V_2$ be a 2-dimensional subspace of $V_3$ with metric, or first fundamental tensor,

$$(2.2) \qquad a_{\alpha\beta} = g_{ij} x^i_\alpha x^j_\beta ,$$

where $x^i(u^1,u^2)$ is a parameterization of $V_2$ , and

$$(2.3) \qquad x^i_\alpha = \frac{\partial x^i}{\partial u^\alpha} .$$

Equation (2.2) requires that $V_2$ be isometrically immersed in $V_3$ . The obvious question is whether under the conformal map, is $\hat{V}_2$ isometrically immersed in $\hat{V}_3$ ?

First, we have

$$(2.4) \qquad \hat{a}_{\alpha\beta} = \hat{g}_{ij} x^i_\alpha x^j_\beta = e^{2\sigma} a_{\alpha\beta}$$

so that $V_2$ and $\hat{V}_2$ are conformally related. N.B. : As in Chapter II in defining the conformal map we impose the same local coordinates on both $V_3$ and $\hat{V}_3$ , i.e., $\hat{x}^i = x^i$ .

Let $V_2$ have unit normal vector, $\underset{\sim}{\xi}$ , with covariant components, $\xi_i$ , and contravariant components, $\xi^i$ . Under the conformal map we have

(2.5) $$\hat{\xi}_i = e^{\sigma}\xi_i ,$$

and

(2.6) $$\hat{\xi}^i = e^{-\sigma}\xi^i .$$

By Gauss's formulae we will deduce the second fundamental tensor $\hat{b}_{\alpha\beta}$ of $\hat{V}_2$. Using the notation of Chapter II to denote covariant differentiation, we have

(2.7) $$\xi^n\hat{b}_{\alpha\beta} = x^n_{\alpha;\beta} = x^n_{\alpha,\beta} - \xi^n\sigma_{/\xi}a_{\alpha\beta} ,$$

(2.8) $$\hat{\xi}^n\hat{b}_{\alpha\beta} = (x^n_{\alpha,\beta} - \xi^n\sigma_{/\xi}a_{\alpha\beta}) ,$$

and

(2.9) $$\hat{\xi}^n\hat{b}_{\alpha\beta} = \xi^n(b_{\alpha\beta} - \sigma_{/\xi}\, a_{\alpha\beta}) = e^{\sigma}\hat{\xi}^n(b_{\alpha\beta} - \sigma_{/\xi}a_{\alpha\beta}) ,$$

where $\sigma_{/\xi} = \sigma_i\xi^i$.

Hence, we obtain

(2.10) $$\hat{b}_{\alpha\beta} = e^{\sigma}(b_{\alpha\beta} - \sigma_{/\xi}a_{\alpha\beta}) .$$

We must now check that the formula of Weingarten holds in $\hat{V}_3$, i.e.,

(2.11) $$\hat{\xi}^n_{;\alpha} = -\hat{a}^{\beta\gamma}\hat{b}_{\alpha\beta}x^n_{\gamma} .$$

By direct calculation we have

$$\hat{\xi}^n{}_{;\alpha} = \hat{\xi}^n{}_{;i} x^i_\alpha = e^{-\sigma}(\xi^n{}_{,i} + \delta^n_i \sigma_{/\xi} - \sigma^n \xi_i) x^i_\alpha \ , \tag{2.12}$$

where $\sigma^n := g^{hj}\sigma_j$ , so that

$$\hat{\xi}^n{}_{;\alpha} = e^{-\sigma}(\xi^n{}_{,\alpha} + \sigma_{/\xi} x^n_\alpha) \ . \tag{2.13}$$

However, we see that

$$\begin{aligned} -\hat{a}^{\beta\gamma}\hat{b}_{\alpha\beta} x^n_\gamma &= -e^{-\sigma} a^{\beta\gamma}(b_{\alpha\beta} - \sigma_{/\xi} a_{\alpha\beta}) x^n_\gamma \\ &= -e^{-\sigma}(\xi^n{}_{,\alpha} +- \sigma_{/\xi} \; x^n_\alpha) \\ &= e^{-\sigma}(\xi^n{}_{,\alpha} + \sigma_{/\xi} \; x^n_\alpha) \ , \end{aligned} \tag{2.14}$$

and thus, Weingarten's equations are satisfied.

We now check that the equations of Gauss and Mainardi-Codazzi hold in $\hat{V}_3$ , i.e.,

$$\hat{R}_{\alpha\beta\gamma\delta} = \hat{b}_{\alpha\gamma}\hat{b}_{\beta\delta} - \hat{b}_{\beta\gamma}\hat{b}_{\alpha\delta} + \hat{R}_{hijk} x^h_\alpha x^i_\beta x^j_\gamma x^k_\delta \ , \tag{2.15}$$

$$\hat{b}_{\alpha\beta;\gamma} - \hat{b}_{\alpha\gamma;\beta} - \hat{R}_{hijk}\hat{\xi}^h x^i_\alpha x^j_\beta x^k_\gamma = 0 \ . \tag{2.16}$$

Equation (2.15) is the Gauss equation and (2.16) are the equations of Mainardi-Codazzi. We verify these in the usual way using (2.2), (2.10), (2.11), and

$$x^n_{\alpha;\beta;\gamma} - x^n_{\alpha;\gamma;\beta} = \hat{R}^\sigma{}_{\alpha\beta\gamma} = \hat{a}^{\sigma\delta} x^n_\delta \hat{R}_{\sigma\alpha\beta\gamma} \ . \tag{2.17}$$

Employing (2.10) and (2.11), (2.17) becomes

$$(2.18) \qquad \hat{a}^{\delta\sigma} x^h_\delta (\hat{R}_{\sigma\alpha\beta\gamma} - (\hat{b}_{\sigma\beta}\hat{b}_{\alpha\gamma} - \hat{b}_{\alpha\beta}\hat{b}_{\sigma\gamma}))$$

$$- \hat{\xi}^h (\hat{b}_{\alpha\beta;\gamma} - \hat{b}_{\alpha\gamma;\beta}) - \hat{R}^h{}_{ijk} x^i_\alpha x^j_\beta s^k_\gamma = 0 \ ,$$

and upon multiplying (2.18) by $\hat{g}_{hj} x^j_\beta$ and contracting on $h$ we obtain

$$(2.19) \qquad \hat{R}_{\alpha\beta\gamma\delta} = \hat{b}_{\alpha\gamma}\hat{b}_{\beta\delta} - \hat{b}_{\beta\gamma}\hat{b}_{\alpha\delta} + \hat{R}_{hijk} x^h_\alpha x^i_\beta x^j_\gamma x^k_\delta \ .$$

Finally, multiplying (2.18) by $\hat{\xi}^h$ and contracting on $h$ we have

$$(2.20) \qquad \hat{b}_{\alpha\beta;\gamma} - \hat{b}_{\alpha\gamma;\beta} - \hat{R}_{hijk}\hat{\xi}^h x^i_\alpha x^j_\beta x^k_\gamma = 0 \ .$$

Thus subspaces act "properly" under conformal maps, viz isometric immersion is preserved under (2.1).

## §3. RICCI ROTATION COEFFICIENTS IN $V_3$ AND $\hat{V}_3$

Let $\{\underset{\sim}{e}_a\}$ denote an orthogonal ennuple, i.e., a set of unit vectors which are orthogonal and have the components $e_a{}^i$ where the index "i" is a contravariant tensor index. Then we have

$$(3.1) \qquad g_{ij} e^i_a e^j_b = 1 \quad \text{if} \quad a = b$$
$$= 0 \quad \text{if} \quad a \neq b \ ,$$

i.e.,

$$(3.2) \qquad g_{ij} e^i_a e^j_b = \delta_{ab} \ .$$

By convention the ennuple index will be written adjacent to the letter denoting the vector. The letters a-g will denote ennuple indices and h-p will denote tensor indices.

Definition 3.1. The Ricci rotation coefficients are given by

$$\gamma_{abc} := e_{ai,j} e_b^i e_c^j \ . \tag{3.3}$$

NB: The $\gamma_{abc}$ are invariants under coordinate transformations but not under ennuple transformations.

The rotation coefficients are interpreted as the rate of rotation of the ennuple vector $\underset{\sim}{e}_a$ with respect to $\underset{\sim}{e}_b$ and $\underset{\sim}{e}_c$. The rotation coefficients are related to, but not identical to the Cartan structural coefficients.

It is easy to see that

$$\gamma_{aab} = 0 \ , \qquad \text{(NS)} \tag{3.4}$$

and

$$\gamma_{abc} + \gamma_{bac} = 0 \ . \tag{3.5}$$

Under a conformal map $V_3 \to \hat{V}_3$ the components of $\{\underset{\sim}{e}_a\}$ transform as

$$\begin{cases} \hat{e}_{ai} = e^{\sigma} e_{ai} \ , \\ \hat{e}_a{}^i = e^{-\sigma} e_a{}^i \ , \end{cases} \tag{3.6}$$

where $\sigma$ is the same function as in (2.1). We now compute the rotation coefficients, $\hat{\gamma}_{abc}$, of the ennuple $\{\hat{e}_a\}$ in $\hat{V}_3$. By direct calculation we have

$$\begin{aligned}
\hat{e}_{ai,j} &= \hat{e}_{ai|j} - \hat{\Gamma}_{ij}{}^k \hat{e}_{ak} \\
&= (e^{\sigma} e_{ai})_{|j} - e^{\sigma} \hat{\Gamma}_{ij}{}^k e_{ak} \\
&= e^{\sigma}(\sigma_j e_{ai} + e_{ai|j} - \Gamma_{ij}{}^k e_{ak} - \delta_i^k \sigma_j e_{ak} \\
&\qquad - \delta_j^k \sigma_i e_{ak} + g_{ij} g^{km} \sigma_m e_{ak}) \\
&= e^{\sigma}(\sigma_j e_{ai} + e_{ai,j} - \sigma_j e_{ai} - \sigma_i e_{aj} + g_{ij} \sigma_m e_a{}^m) \\
&= e^{\sigma}(e_{ai,j} - \sigma_i e_{aj} + g_{ij} \sigma_{/a}) ,
\end{aligned}$$

where $\sigma_{/a} := \sigma_i e_a^i$. Thus, we have

$$\hat{e}_{ai;j} = e^{\sigma}(e_{ai,j} - \sigma_i e_{aj} + g_{ij} \sigma_{/a}) . \tag{3.7}$$

Moreover, we have the equation

$$\hat{e}_a{}^i{}_{;j} = e^{-\sigma}(e_a{}^i{}_{,j} - \sigma^i e_{aj} + \delta^i_j \sigma_{/a}) \tag{3.8}$$

by virtue of

$$\hat{g}_{ij;k} = 0 , \tag{3.9}$$

and

$$\hat{g}^{ij}{}_{;k} = 0 . \tag{3.10}$$

Hence, we find that

(3.11) $$\hat{\gamma}_{abc} = \hat{e}_{ai;j}\hat{e}_b{}^i\hat{e}_c{}^j = e^{-\sigma}(\gamma_{abc} - \sigma_{/b}\delta_{ac} + \sigma_{/a}\delta_{bc}) \ ,$$

and

(3.12) $$\hat{\gamma}_{abc} = e^{-\sigma}\gamma_{abc} \ ,$$

when a, b, c are distinct.

We now give a definition that we will need in the rest of the Chapter.

Definition 3.2. A **congruence of curves** in $V_N$ is a family of curves such that one of the curves of the family passes through each point of a chart of $V_N$ .

If $\underset{\sim}{\xi}$ is a vector field defined on chart of $V_N$ , then the integral curves of $\underset{\sim}{\xi}$ define a congruence of curves and the vectors $\underset{\sim}{\xi}$ are the tangent vectors to the curves of the congruence.

## §4. OTHER CRITERIA FOR CONFORMAL FLATNESS[1)]

In Chapter II, §5, we showed that for $N > 3$ , $V_N$ is a $C_N$ if and only if the Weyl tensor $C_{hijk}$ vanishes; and that for $N \geq 3$ $V_N$ is a $C_N$ if and only if the Finzi tensor $L_{ijk}$ vanishes. In this

[1]A version of the material in this section has been submitted to Tensor, N.S.

section we will give criteria for conformal flatness involving orthogonal ennuples.

**Theorem 4.1** (Schouten-Eisenhart) SCHOUTEN[1921] and EISENHART[1949]. $V_N$ is a $C_N (N > 3)$ if and only if for every orthogonal ennuple $\{\underset{\sim}{e}_a\}$

$$(4.1) \qquad R_{abcd} := R_{hijk} e_a{}^h e_b{}^i e_c{}^j e_d{}^k = 0$$

for distinct values of $a, b, c, d = 1, 2, \ldots, N$ .

Proof. The first half is easy. If $V_N$ is a $C_N$ then $C_{hijk} = 0$ so that

$$(4.2) \qquad C_{abcd} := C_{hijk} e_a{}^h e_b{}^i e_c{}^j e_d{}^k = 0 ,$$

and we have

$$(4.3) \qquad R_{abcd} = - \frac{1}{N-2}(g_{ac}R_{bd} - g_{ad}R_{bc} + g_{bd}R_{ac} - g_{bc}R_{ad}) - \frac{R}{(N-2)(N-1)}(g_{ac}g_{bd} - g_{ad}g_{bc}) ,$$

and for an orthogonal ennuple we have $g_{ab} = \delta_{ab}$ . Thus we have that $R_{abcd} = 0$ for distinct values of $a, b, c, d$ .

To establish the converse we must first exhibit the ennuple components of the Weyl tensor. For distinct values of $a, b, c, d$ these are

$$(4.4) \qquad C_{abcd} = R_{abcd} ,$$

$$C_{abad} = R_{abad} + \frac{1}{N-2} R_{bd} , \tag{4.5}$$ (NS)

$$C_{abab} = R_{abab} + \frac{1}{N-2} (R_{bb} + R_{aa}) - \frac{R}{(N-1)(N-2)} , \tag{4.6}$$ (NS)

where

$$R_{ab} := R_{ij} e_a{}^i e_b{}^j . \tag{4.7}$$

Hence, we have the equations

$$R_{abcd} = 0 , \tag{4.8}$$

$$R_{abad} = - \frac{1}{N-2} R_{bd} , \tag{4.9}$$ (NS)

$$R_{abab} = - \frac{1}{N-2} (R_{bb} + R_{aa}) + \frac{R}{(N-1)(N-2)} . \tag{4.10}$$ (NS)

These are easily seen to be equivalent to

$$R_{abcd} = 0 , \tag{4.11}$$

$$R_{abad} = R_{cbcd} , \tag{4.12}$$ (NS)

$$R_{abab} + R_{cdcd} = R_{acac} + R_{bdbd} \tag{4.13}$$ (NS)

from which it follows that all the components of $C_{abcd}$ are zero.

We now give a more geometric criterion for $V_N$ to be a $C_N$ . This was first established by RICCI[1918] for $N = 3$ and generalized by FINZI[1922] for $N > 3$ .

**Theorem 4.2** (Ricci-Finzi). $V_N$ is a $C_N$, $(N \geq 3)$, if and only if there exists an orthogonal ennuple of $\{e_a\}$ such that the correspondence congruences of curves $\{l_a\}$ (curves having tangent vectors $e_a$) are **normal** and **isotropic**, i.e., their rotation coefficients $\gamma_{abc}$ satisfy the respective conditions:

$$\gamma_{abc} = 0 , \tag{4.14}$$

$$\gamma_{abb} = \gamma_{acc} \qquad \text{(NS)} \tag{4.15}$$

for distinct values of $a, b, c = 1, 2, \ldots, N$.

The condition (4.14) for normality imposes $N(N - 1)(N - 2)/2$ conditions on the $\gamma_{abc}$ and has the usual meaning, viz. that there exists an N-tuply orthogonal system of (hyper)surfaces $\{\Sigma_a\}$ orthogonal to the respective congruences $\{l_a\}$. The isotropy conditions (4.15) impose $N(N - 2)$ additional conditions on the $\gamma_{abc}$ and by the familiar geometric interpretation of the rotation coefficients this means that each $l_a$ is turning, i.e., "rotating", at the same rate in each of the directions $e_b$ $(b \neq a)$ specified by the other congruences $l_b$. Thus (4.14) and (4.15) reduce the number of independent rotation coefficients from $N^2(N - 1)/2$ to $N$.

Ricci proved this for $N = 3$ by directly integrating the system of equations given by (4.14) and (4.15) and noting that the resulting space was a $C_3$. For $N > 3$, Finzi observed that (4.14) and (4.15), when substituted in the rotation coefficient expressions for $R_{abcd}$,

$$R_{abcd} := \gamma_{abc/d} - \gamma_{abd/c} + \sum_f (\gamma_{abf}(\gamma_{fcd} - \gamma_{fdc}) + \gamma_{fad}\gamma_{fbc} - \gamma_{fac}\gamma_{fbd}) , \tag{4.16}$$

led to equations (4.11), (4.12) and (4.13).

We now prove a Schouten-Eisenhart like theorem for $N = 3$ .

**Theorem 4.3.** $V_3$ is a $C_3$ if and only if for every orthogonal ennuple $\{\underset{\sim}{e}_a\}$

$$L_{abc} := L_{ijk} e_a{}^i e_b{}^j e_c{}^k = 0 \tag{4.17}$$

for distinct values of $a, b, c = 1, 2, 3$ .

Proof. Let $\{\underset{\sim}{e}_a\}$ be an orthogonal ennuple and define another orthogonal ennuple $\{e_{a'}\}$ by

$$\begin{aligned} \underset{\sim}{e}_{a'} &= \mu \underset{\sim}{e}_a + \nu \underset{\sim}{e}_b , \\ \underset{\sim}{e}_{b'} &= -\nu \underset{\sim}{e}_a + \mu \underset{\sim}{e}_b , \\ \underset{\sim}{e}_{c'} &= \underset{\sim}{e}_c \end{aligned} \tag{4.18}$$

for arbitrary values of the scalars $\mu$ and $\nu$ . Then by (4.17) we have

$$L_{a'b'c'} := L_{ijk} e_{a'}{}^i e_{b'}{}^j e_{c'}{}^k = 0 , \tag{4.19}$$

and by using (4.17) we must have

(4.20) $$\mu\nu(L_{bbc} - L_{aac}) = 0 \,. \qquad \text{(NS)}$$

Thus, since $\mu$ and $\nu$ are arbitrary it follows that

(4.21) $$L_{aac} = L_{bbc} \qquad \text{(NS)}$$

for distinct values of $a, b, c$. However, (4.21) holds for $a, b \neq c$ and upon summing over $a$ we have

(4.22) $$\sum_a L_{aac} = 2L_{bbc} \,, \qquad \text{(NS on } b)$$

but the left-hand side is identically zero by the doubly contracted differential Bianchi identity. Thus $L_{abc} = 0$ for distinct values of $a, b, c$ implies $L_{aac} = 0$ for $a \neq c$. Therefore, $L_{abc} = 0$ for distinct values of $a, b, c$ requires that the space $V_3$ be conformally flat. The converse is obvious and our proof is complete.

The relative effectiveness of the Schouten-Eisenhart and Ricci-Finzi criteria can be easily seen. Suppose $V_N (N \geq 3)$ is given and it is required to determine whether $V_N$ is a $C_N$. By using the Schouten-Eisenhart criteria (Theorem 4.1 or 4.3) we would choose an orthogonal ennuple $\{e_a\}$ and test for $R_{abcd} = 0$ or $L_{abc} = 0$. If $R_{abcd} \neq 0$ or $L_{abc} \neq 0$, we are done and $V_N$ is not a $C_N$. But if $R_{abcd} = 0$ or $L_{abc} = 0$ for $\{e_a\}$, or even for several different ennuples, there is no guarantee that $V_N$ is a $C_N$ since the theorem requires that this must hold for **every** orthogonal ennuple. The Schouten-Eisenhart criteria can easily determine whether $V_N$ is not a $C_N$ by a **single** *choice of ennuple, but in practice* they are inadequate to determine whether $V_N$ is a $C_N$.

On the other hand, the Ricci-Finzi criterion proceeds by attempting to solve the system of partial differential equations (4.14) and (4.15) for the $\{\underset{\sim}{e}_a\}$. Upon converting these to $\lambda$-coefficients (in the notation used by LANDAU/LIFSHITZ[1975])

(4.23) $$\lambda_{abc} := \gamma_{abc} - \gamma_{acb} ,$$

where

(4.24) $$\gamma_{abc} = (\lambda_{abc} + \lambda_{bca} - \lambda_{cab})/2$$

one can employ partial derivatives, and avoid computing Christoffel symbols of $V_N$. Equations (4.14) and (4.15) are equivalent to

(4.25) $$\lambda_{abc} = 0 ,$$

(4.26) $$\lambda_{bba} = \lambda_{cca} \qquad \text{(NS)}$$

for distinct values of $a, b, c = 1, 2, \ldots, N$. If a solution of these equations can be found, then $V_N$ is a $C_N$. Moreover, the Ricci-Finzi criterion then produces the "nicest" ennuple $\{\underset{\sim}{e}_a\}$, i.e., one which is directly tied to the geometry of $V_N$, and when $V_N$ is a $C_N$ a solution of these equations is guaranteed.

## §5. NORMAL, GEODESIC, AND CANONICAL CONGRUENCES.

**Definition 5.1.** A congruence of curves $\Gamma$ in $V_3$ is **normal** if the unit tangent vectors to the curves of the congruence are the unit normals of a family of surfaces.

Hence, if $\xi_i$ are the components of the unit tangent vectors of a normal congruence of curves, then there exists a function $\phi : V_3 \to \mathbb{R}$ such that

$$(5.1) \qquad \phi_{,i} = \phi_{|i} = \phi_i = \psi \xi_i \; .$$

In other words, the unit tangent vectors of a normal congruence are proportional to a gradient.

Suppose that $\{\underset{\sim}{e}_a\}$ is an orthogonal ennuple in $V_3$. The conditions for the congruence $l_a$ defined by $\underset{\sim}{e}_a$ to be normal are well-known to be

$$(5.2) \qquad \gamma_{abc} - \gamma_{acb} = 0 \; ,$$

where $b \neq c$ and $b, c \neq a$. See EISENHART[1949], p. 115.

**Definition 5.2.** A <u>geodesic</u> congruence of curves in $V_3$ is one such that each of the curves of the congruence is a geodesic. That is, if $\xi_i$ are the components of the unit tangent vectors of a geodesic congruence, then

$$(5.3) \qquad \xi_{i,j}\xi^j = 0 \; .$$

If $\{\underset{\sim}{e}_a\}$ is an orthogonal ennuple in $V_3$, then the conditions that $\underset{\sim}{e}_a$ defines a geodesic congruence $l_a$ are

$$(5.4) \qquad \gamma_{baa} = 0 \; , \qquad \text{(NS)}$$

where $b = 1, 2, 3$. See EISENHART[1949], p. 100. For $\underset{\sim}{e}_a$ to define a congruence of curves that is simultaneously normal and geodesic it is *necessary and sufficient* that

$$(5.5) \qquad e_{ai,j} = e_{aj,i} .$$

See EISENHART[1949], p. 117.

Suppose that $\underset{\sim}{\xi}$ defines a congruence of curves denoted by $l_3$[2]. We will construct two new congruences $l_1$ and $l_2$ with unit tangent vectors $\underset{\sim}{\lambda}$ and $\underset{\sim}{\rho}$, respectively, using the method of RICCI[1918].

Let $\xi_i$ be the components of $\underset{\sim}{\xi}$ and define

$$(5.6) \qquad X_{i,j} := \frac{1}{2}(\xi_{i,j} + \xi_{j,i}) .$$

Consider the system of equations

$$(5.7) \qquad \begin{cases} \xi_i \zeta^i = 0 , \\ (X_{ij} - \omega g_{ij})\zeta^i + \mu \xi_j = 0 , \end{cases}$$

where $\omega$ and $\mu$ are scalars and $\zeta^i$ are the components of a vector. The equations (5.7) have the determinant

[2]In the following discussion it is convenient to number the congruences in a manner which corresponds to our labelling of the ennuple vectors $\{\underset{\sim}{e}_a\}$.

$$
(5.8) \qquad \begin{vmatrix} X_{11} - \omega g_{11} & X_{12} - \omega g_{12} & X_{13} - \omega g_{13} & \xi_1 \\ X_{12} - \omega g_{12} & X_{22} - \omega g_{22} & X_{23} - \omega g_{23} & \xi_2 \\ X_{13} - \omega g_{13} & X_{23} - \omega g_{23} & X_{33} - \omega g_{33} & \xi_3 \\ \xi_1 & \xi_2 & \xi_3 & 0 \end{vmatrix} = 0
$$

which yields a second-degree polynomial in $\omega$. The roots of this polynomial are real and when they are inserted into (5.7) determine two real congruences of curves $\Gamma_1$ and $\Gamma_2$ having unit tangent vectors $\lambda$ and $\rho$, respectively. If the roots are unique, then $\lambda$ and $\rho$ are uniquely determined. Regardless of whether $\lambda$ and $\rho$ are unique, they are said to be <u>canonical</u> with respect to $\xi$, and $\Gamma_1$ and $\Gamma_2$ are said to be <u>canonical</u> with respect to $\Gamma_3$. An example of the calculation of canonical congruences is given in the appendix.

We now determine conditions on the rotation coefficients of an orthogonal ennuple $\{e_a\}$ when the vectors of the subset $\{e_b\}$ $(b \neq a)$ are canonical with respect to $e_a$. From (3.3) it is easy to see that

$$
(5.9) \qquad e_{ai,j} = \sum_{f,g} [illegible] (e_{fi} e_{gj}) .
$$

Applying the definition of $X_{ij}$ [illegible] obtain

$$
(5.10) \qquad \frac{1}{2} \sum_d (\gamma_{abd} + \gamma_{adb}) [illegible] e_{bj} + [illegible] e_{aj} = 0 ,
$$

where $\omega_b$ is the root of (5.8) corresponding to $\underset{\sim}{e}_b$ and $\mu_b$ is the scalar from (5.7) corresponding to $\underset{\sim}{e}_b$. Multiplying (5.10) by $e_c{}^j$ with $c \neq a$, $b$ and contracting on $j$ we have

$$\gamma_{abc} + \gamma_{acb} = 0 , \tag{5.11}$$

for $b \neq c$ and $a \neq b , c$. Equation (5.10) also implies

$$\omega_b = \gamma_{abb} , \qquad \text{(NS)} \tag{5.12}$$

and

$$\mu_b = \frac{1}{2} \gamma_{baa} , \qquad \text{(NS)} \tag{5.13}$$

where $a \neq b$.

In the following we will need to know conditions on the rotation coefficients of the orthogonal ennuple $\{\underset{\sim}{e}_a\}$ when $\underset{\sim}{e}_a$ defines a normal congruence and the vectors of the subset $\{\underset{\sim}{e}_a\}$ $(b \neq a)$ are canonical with respect to $\underset{\sim}{e}_a$. Combining (5.2) and (5.11) we have, in this case,

$$\gamma_{abc} = 0 , \tag{5.14}$$

$b \neq c$ and $a \neq b , c$. Rodrigues's formula McCONNELL[1931], p. 216 and (5.14) imply that the vectors $\underset{\sim}{e}_b$ $(b \neq a)$ are principal directions on the surfaces normal to $\underset{\sim}{e}_a$. Moreover, $\omega_b$ $(b \neq a)$ are the principal curvatures of the directions $\underset{\sim}{e}_b$ $(b \neq a)$ and

$\frac{1}{2}\mu_b (b \neq a)$ are the components of the curvature vector of $\underset{\sim}{e}_a$ in the subset $\{\underset{\sim}{e}_b\}$ $(b \neq a)$ .

We now have the machinery to give our promised proof of Theorem 1.2 of Chapter III, the generalized Dupin theorem[3].

We prove only sufficiency since the necessity was proved in Chapter III. Choose two of the families of surfaces and call them $\Sigma_1$ and $\Sigma_2$ . Let $\xi_i$ be the components of the unit normal $\underset{\sim}{\xi}$ to $\Sigma_1$ , and denote by $\Gamma_3$ the congruence of curves defined by $\xi$ . We employ Ricci's method to construct two congruences $\Gamma_1$ and $\Gamma_2$ with unit tangent vectors $\underset{\sim}{\lambda}$ and $\underset{\sim}{\rho}$ , respectively, that are canonical with respect to $\Gamma_3$ . Since $\underset{\sim}{\lambda}$ and $\underset{\sim}{\rho}$ are principal directions on $\Sigma_1$ , and $\Sigma_1$ and $\Sigma_2$ intersect in a line of curvature, then either $\underset{\sim}{\lambda}$ or $\underset{\sim}{\rho}$ is a principal direction on $\Sigma_2$ . Without loss of generality we may choose it to be $\underset{\sim}{\lambda}$ . Since $\Sigma_1$ and $\Sigma_2$ are orthogonal, $\underset{\sim}{\xi}$ must lie in the tangent plane to $\Sigma_2$ , and since $\underset{\sim}{\xi}$ is perpendicular to $\underset{\sim}{\lambda}$ , it is a principal direction on $\Sigma_2$ . Moreover, $\underset{\sim}{\rho}$ is perpendicular to both $\underset{\sim}{\lambda}$ and $\underset{\sim}{\xi}$ , and hence must be the unit normal to $\Sigma_2$ . But since $\underset{\sim}{\xi}$ and $\underset{\sim}{\lambda}$ are principal directions on $\Sigma_2$ , they are canonical with respect to $\underset{\sim}{\rho}$ and by construction $\underset{\sim}{\lambda}$ and $\underset{\sim}{\rho}$ are canonical with respect to $\underset{\sim}{\xi}$ . If we

[3]A version of the following material will appear in Tensor, N.S. See MOORE ZUND[1986].

label our ennuple $\{\underset{\sim}{e}_a\}$ in the order $\{\underset{\sim}{\lambda}, \underset{\sim}{\rho}, \underset{\sim}{\xi}\}$, we may express this in terms of rotation coefficients as

$$\gamma_{3ab} = 0 , \tag{5.15}$$

where $a \neq b$ and $a, b \neq 3$, and

$$\gamma_{2ab} = 0 , \tag{5.16}$$

where $a \neq b$ and $a, b \neq 2$. But by the skew-symmetry of the rotation coefficients in the first two indices, (5.15) and (5.16) imply that

$$\gamma_{abc} = 0 \tag{5.17}$$

for distinct values of $a, b, c$. However, (5.17) are necessary and sufficient for conditions that **all** the congruences of curves associated with the ennuple are normal. Thus $\Gamma_1$ is a normal congruence. Hence, there exist a third family of surfaces $\Sigma_3$ orthogonal to both $\Sigma_1$ and $\Sigma_2$. This completes the proof.

Finally, we examine how canonical and normal congruences behave under conformal maps.

**Theorem 5.1**. If $V_3$ and $\hat{V}_3$ are conformally related and $\{\underset{\sim}{\lambda}, \underset{\sim}{\rho}, \underset{\sim}{\xi}\}$ is an orthogonal ennuple in $V_3$ with $\underset{\sim}{\lambda}$ and $\underset{\sim}{\rho}$ canonical with respect to $\underset{\sim}{\xi}$, then the conformal image of this ennuple in $\hat{V}_3$ is defined by

$$\hat{\lambda}_i = e^{\sigma}\lambda_i \,,$$

$$(5.18) \qquad \hat{\rho}_i = e^{\sigma}\rho_i \,,$$

$$\hat{\xi}_i = e^{\sigma}\xi_i \,,$$

and $\hat{\underset{\sim}{\lambda}}$ , $\hat{\underset{\sim}{\rho}}$ are canonical with respect to $\hat{\underset{\sim}{\xi}}$ .

Proof. Let $\gamma_{abc}$ and $\hat{\gamma}_{abc}$ be the rotation coefficients of the respective ennuples in $V_3$ and $\hat{V}_3$ . From (3.12) we have that

$$(5.19) \qquad \hat{\gamma}_{abc} = e^{2\sigma}\gamma_{abc}$$

for distinct values of a, b, c . Since $\underset{\sim}{\lambda}$ and $\underset{\sim}{\rho}$ are canonical with respect to $\underset{\sim}{\xi}$ we have

$$(5.20) \qquad \gamma_{312} + \gamma_{321} = 0 \,,$$

and (5.19) implies that

$$(5.21) \qquad \hat{\gamma}_{312} + \hat{\gamma}_{321} = 0 \,.$$

Thus $\hat{\underset{\sim}{\lambda}}$ and $\hat{\underset{\sim}{\rho}}$ are canonical with respect to $\hat{\underset{\sim}{\xi}}$ .

**Theorem 5.2**. Let $V_3$ and $\hat{V}_3$ be conformally related. If $\Gamma_3$ is a normal congruence of curves in $V_3$ with unit tangent vectors $\underset{\sim}{\xi}$ , then the conformal image $\hat{\Gamma}_3$ is a normal congruence of curves in $\hat{V}_3$ with unit normal $\hat{\underset{\sim}{\xi}}$ where $\hat{\underset{\sim}{\xi}}$ is defined by

(5.22) $$\hat{\xi}_i = e^{\sigma}\xi_i \;.$$

Proof. Use Ricci's method to construct congruences $\Gamma_1$ and $\Gamma_2$ with unit tangent vectors $\lambda$ and $\rho$, respectively, which are canonical with respect to $\Gamma_3$. Then we know that

(5.22) $$\gamma_{312} = \gamma_{321} = 0 \;,$$

and from (3.12) we have that $\hat{\gamma}_{abc} = e^{-\sigma}\gamma_{abc}$ for a, b, c distinct. Thus

(5.23) $$\hat{\gamma}_{312} = \hat{\gamma}_{321} = 0$$

and $\hat{\xi}$ is the unit tangent vector of a normal congruence $\hat{\Gamma}_3$ in $\hat{V}_3$.

## §6. A ROTATION COEFFICIENT FORMULATION OF THE CAYLEY-DARBOUX EQUATION

Let $\xi$ define a normal congruence of curves in $V_3$. If $\lambda$ and $\rho$ are canonical with respect to $\xi$ then

(6.1) $$\gamma_{312} = \gamma_{321} = 0 \;,$$

where $\gamma_{abc}$ are the rotation coefficients of $\{\lambda, \rho, \xi\}$. The condition that all the congruences of an ennuple be normal is

(6.2) $$\gamma_{abc} = 0$$

for distinct values of $a, b, c$ . The skew-symmetry of the rotation coefficients in the first two indices means (6.2) will hold when (6.1) holds and that

$$(6.3) \qquad \gamma_{123} := \lambda_{i,j}\rho^i\xi^j = 0 \ .$$

Since $\underset{\sim}{\xi}$ defines a normal congruence, there exist $\phi : V_3 \to \mathbb{R}$ such that

$$(6.4) \qquad \phi_i = \varphi\xi_i \ .$$

If (6.1) is given, then (6.3) implies that there exist $\Psi$ , $\theta : V_3 \to \mathbb{R}$ such that

$$(6.5) \qquad \begin{cases} \Psi_i = \psi\lambda_i \ , \\ \theta_i = \theta\rho_i \ . \end{cases}$$

Thus $\{\phi,\Psi,\theta\}$ is a triply orthogonal system of coordinates in $V_3$ . Hence, when (6.1) is given, (6.3) is equivalent to the Cayley-Darboux equation of Chapter III, §3.

## §7. HOTINE'S CONJECTURE

As stated in the introduction to this Chapter, Martin Hotine conjectured and claimed to prove that any function $\phi : E_3 \to \mathbb{R}$ with non-vanishing gradient could be a member of a triply orthogonal coordinate system. We quote from HOTINE[1966b], pp. 196-198, with minor changes to agree with the notation and numbering of this dissertation.

"In HOTINE[1966a], it was shown that the gradient equation of a scalar $\phi$ ,

$$\phi_i = \varphi\xi_i \tag{7.1}$$

($\xi_i$ a unit vector), can be transformed conformally with scale factor $\varphi[= e^{\sigma}]$ to a Riemannian space in which the $\xi_i$ become tangents to a family of geodesics and the $\phi$-surfaces, that is the surfaces over which $\phi$ is constant, become geodesic parallels. It followed that the metric of the curved Riemannian space (denoted by hats) can be written in the geodesic form,

$$d\hat{s}^2 = \hat{a}_{\alpha\beta}dx^{\alpha}dx^{\beta} + d\phi^2 \qquad (\alpha,\beta = 1,2) \tag{7.2}$$

(EISENHART[1949], p. 57; WEATHERBURN[1938], p.81). Using the same coordinates $(x^{\alpha},\phi)$ the metric of the untransformed space can accordingly be written as

$$ds^2 = a_{\alpha\beta}dx^{\alpha}dx^{\beta} + e^{-2\sigma}d\phi^2 \qquad (\alpha,\beta = 1,2) . \tag{7.3}$$

"The components of $a_{\alpha\beta}$ can of course contain $\phi$ , but for different constant values of $\phi$ will also be the surface metrics of the $\phi$-surfaces. If $|a|$ is the determinant of the metric of the $\phi$-surface passing through a point, then it is clear from (7.3) that the determinant of the three-dimensional metric at that point is

$$e^{-2\sigma}|a| . \tag{7.4}$$

Consequently the associated tensor in three-dimensions is

$$(7.5) \qquad g^{rs} = (a^{\alpha\beta}, e^{2\sigma}) \qquad (r,s = 1,2,3)$$

in which $a^{\alpha\beta}$ is the associated tensor of the surface. This can easily be verified by writing out the metric tensor in full.

"Now any curvilinear coordinate system in three-dimensions implies the existence of three scalars, or coordinates, whose gradient vectors are not coplanar. A coordinate line may be defined as a line along which only one of the scalars varies, the other two being constant. Each coordinate line must accordingly be perpendicular to the gradient vectors of the other two coordinates. The $x^1$-coordinate line is perpendicular to the gradient of $\phi$, which from (7.1) is in the direction $\xi_r$ normal to the $\phi$-surface, so that the $x^1$-coordinate line (and similarly, the $x^2$-coordinate line) must lie in the $\phi$-surface, and $(x^1,x^2)$ can therefore be considered as surface as well as space coordinates.

It is apparent from the absence of $(\alpha,3)$ components in the metric (7.3) that the $\phi$-coordinate line is perpendicular to both the $x^1$- and $x^2$-coordinate lines, so that $\xi_r$ is the direction of both the $\phi$-coordinate line and the gradient of $\phi$. Consequently the gradient vector of $x^1$, considered as a scalar must lie in the surface, because it must be perpendicular to both the $x^2$- and $\phi$-coordinate lines. The gradient vector of $x^2$ must similarly lie in the surface. We cannot yet say however that the $x^1$- and $x^2$-coordinate lines are orthogonal or that coordinates can be found which would make them orthogonal within the framework of the space metric (7.3).

"We assume nevertheless that the coordinates $(x^1, x^2)$ in the metric (7.3) are definable as scalar functions of position throughout some region of flat space, in other words that they can be expressed as some function of Cartesian coordinates $(x, y, z)$ independently of the definition of $N$, in accordance with the usual meaning of coordinates in a Riemannian flat space. In that case we can write

$$Cp_r = (x^1)_r$$

where $p_r$ is the unit vector in the direction of the gradient of $x^1$ and $C$ is the modulus of the gradient. Evaluating $C$ in the metric (7.3) gives

$$C^2 = g^{rs}(Cp_r)(Cp_s) = g^{rs}(x^1)_r(x^1)_s = g^{11} = a^{11} ,$$

and finally

$$(x^1)_r = \sqrt{a^{11}}\, p_r . \tag{7.6}$$

But $p_r$ is the unit normal to the $x^1$-surface passing through the point under consideration, that is the surface over which $x^1$, considered simply as a scalar, is constant. Equation (7.6) is accordingly in all respects similar to (7.1). By making a conformal transformation to another curved space with conformal factor $a^{11}$, the $p_r$ will transform to geodesics, and exactly as in (2) we can write the metric of this second curved space in the geodesic form

$$d\hat{s}^2 = \hat{a}_{\gamma\delta}\, d\hat{x}^\gamma d\hat{x}^\delta + (dx^1)^2 .$$

Transforming back to the original flat space, we have

$$(7.7) \qquad ds^2 = \frac{\hat{\hat{a}}_{\gamma\delta}}{a^{11}} d\hat{x}^\gamma d\hat{x}^\delta + \frac{1}{a^{11}} (dx^1)^2$$

as the metric of the flat space. We do not yet know what the other two coordinates $d\hat{x}^2$ may be in this metric. The two metrics (7.3) and (7.7) are however alternative ways of expressing the same space and one must transform into the other.

"Since $x^1$ is a scalar function of position in space, it gives rise to a family of surfaces whose metric can be expressed in the following forms putting $dx^1 = 0$ in (7.3) and (7.7)

$$ds^2 = a_{22}(dx^2)^2 + e^{-2\sigma} d\phi^2$$

or

$$ds^2 = \frac{\hat{\hat{a}}_{\gamma\delta}}{a^{11}} d\hat{x}^\gamma d\hat{x}^\delta .$$

These two invariant forms of the line element at a point of the surface hold not only over one particular surface but also over the whole family, so that the coordinates $\hat{\hat{x}}^\gamma$ are either the same as $(x^2,\phi)$ or can be transformed to $(x^2,\phi)$. We can accordingly rewrite (7.7) as

$$(7.8) \qquad ds^2 = a_{22}(dx^2)^2 + \frac{1}{a^{11}} (dx^1)^2 + e^{-2\sigma} d\phi^2 ,$$

which is triply orthogonal in the coordinates $(x^1, x^2, \phi)$ . Comparing this with (7.3) we have $a_{12} = 0$ , which must be so since the coordinates are orthogonal; and $a_{11} = \frac{1}{a^{11}}$ , which is so since $a_{12} = 0$ .

"We get the same result by considering the gradient of the other coordinate $x^1$ .

"In aiming at this result, we have merely assumed that the coordinates $(x^1, x^2)$ in the metric (3) are scalar functions of position, without otherwise restricting them or the form of $\phi$ . Classical doctrine on the subject asserts nevertheless that $\phi$ must satisfy a third-order partial differential equation known to Eisenhart and others as the (Cayley)-Darboux equation. If the above reasoning is correct, then *the (Cayley)-Darboux equation, which is* shown below to be equivalent to one of the six conditions of flat space, must be an identity, in which case it expresses a relation between $\varphi$ and the form of the $\phi$-surfaces. In the main geodetic application, this would be a hitherto unsuspected relation between gravity and the form of the equipotential surfaces. It is accordingly of considerable importance to resolve this question one way or the other.

"The remainder of this paper assumes the existence of a triply orthogonal system derived from a scalar $\phi$ , and on that assumption works out its properties, including several which do not seem to have been formulated before, at any rate in the compact form now given. If the classical view is correct, these results are valid if $\phi$

satisfies the (Cayley)-Darboux equation."

In the next section we will analyze Hotine's conjecture employing the rotation coefficients formalism introduced in §6.

## §8. CONFORMAL MAPPING AND THE CAYLEY-DARBOUX EQUATION

Antonio Marussi was aware of the weaknesses of Hotine's argument and makes the following comment in MARUSSI[1985],[4)] p. 133. We quote it using our notation.

"We now establish an extremely important fact. The change in curvature $e^{\sigma}\varepsilon_i$ depends only on the position and on the direction of the normal to the surface, and not on the direction of the section on it. Since it is the same for all these directions, it follows therefore that the directions of principal curvature are conserved in the representation. Thus, if a family of surfaces is not of Lamé's type in $V_3$, i.e., it does not belong to a triply-orthogonal system, then neither can it transform in $\hat{V}_3$. Since the family of equipotential surfaces of the Earth's gravity field is not of Lamé type (for it to be so, (Cayley)-Darboux's third order partial differential equation would need to be satisfied), then neither can be any of its transforms in a conformal representation; there is thus no possibility of reducing the study of the Earth's potential field into a triply orthogonal coordinate system."

We will state the above argument in more precise mathematical

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[4]This article was originally published in 1967.

form and prove it.

**Theorem 8.1.** Let $V_3$ and $\hat{V}_3$ be conformally related. Let $\Gamma$ denote a congruence of curves in $V_3$. If $\Gamma$ is not a normal congruence, then its conformal image $\hat{\Gamma}$ in $\hat{V}_3$ cannot be a normal congruence.

Proof. The theorem follows by using Ricci's method to find two congruences canonical with respect to $\Gamma$ and by using equation (3.12) and the rotation coefficient criteria in §5 for a congruence to be a normal congruence. See the proof of Theorem 5.2.

Theorem 8.1 and our derivation of the Cayley-Darboux equation in Chapter III, §2 show that Hotine's conjecture is false. In §11, we will further critically analyze Hotine's argument.

## §9. CONDITIONS FOR THE CONFORMAL IMAGE OF A NORMAL CONGRUENCE TO BE A GEODESIC NORMAL CONGRUENCE

In this section we examine conditions on the conformal function $\sigma$ that determine whether the conformal image of a family of surfaces in $V_3$ can be a system of geodesic parallels in $\hat{V}_3$. (A system of geodesic parallels is a family of surfaces such that the unit normals to the surfaces define a geodesic normal congruence of curves.)

**Theorem 9.1.** Let $V_3$ and $\hat{V}_3$ be conformally related and let $\sigma$ be the conformal function. If $\phi : V_3 \to \mathbb{R}$ is a smooth function and we define a vector field $\underset{\sim}{\xi}$ by

$$\phi_i = \varphi\xi_i \tag{9.1}$$

with $\varphi > 0$ , then $\underset{\sim}{\xi}$ defines a normal congruence of curves, viz. the curves normal to the family of surfaces $\phi$ = constant. If $\underset{\sim}{\lambda}$ and $\underset{\sim}{\rho}$ are canonical with respect to $\underset{\sim}{\xi}$ , then the necessary and sufficient conditions that the conformal image of $\underset{\sim}{\xi}$ , $\hat{\underset{\sim}{\xi}}$ , defines a system of geodesic parallels are given by

$$\begin{cases} \varphi_{/\lambda} - \varphi\sigma_{/\lambda} = 0 \, , \\ \varphi_{/\rho} - \varphi\sigma_{/\rho} = 0 \, , \end{cases} \tag{9.2}$$

where $\varphi_{/\lambda} := \varphi_i\lambda^i$ , $\varphi_{/\rho} := \varphi_i\rho^i$ , $\sigma_{/\lambda} := \sigma_i\lambda^i$ and $\sigma_{/\rho} := \sigma_i\rho^i$ .

Proof. From (9.1) we have

$$\phi_{i,j} = \varphi_j\xi_i + \varphi\xi_{i,j} \, , \tag{9.3}$$

but $\phi_{i,j}$ is symmetric in $i$ and $j$ so

$$\varphi(\xi_{i,j} - \xi_{j,i}) = (\varphi_i\xi_j - \varphi_j\xi_i) \, . \tag{9.4}$$

Multiplication by $\xi^j$ and contraction on $j$ yields

$$\sigma_{i,j}\xi^j = \frac{1}{\varphi}(\varphi_i - \varphi_{/\xi}\xi_i) \tag{9.5}$$

where $\varphi_{/\xi} = \varphi_i\xi^i$ .

The conformal image of $\underset{\sim}{\xi}$ , $\hat{\underset{\sim}{\xi}}$ , is defined by

$$(9.6) \qquad \hat{\xi}_i = e^{\sigma}\xi_i \ ,$$

and a straightforward computation shows that

$$(9.7) \qquad \hat{\xi}_{i;j}\hat{\xi}^j = \xi_{i,j}\xi^j - \sigma_i + \xi_i\sigma_{/\xi} \ .$$

Combining (9.7) and (9.5) gives

$$(9.8) \qquad \varphi\hat{\xi}_{i;j}\hat{\xi}^j = \varphi_i - \varphi_{/\xi}\xi_i - \varphi(\sigma_i - \xi_i\sigma_{/\xi}) \ .$$

If we write $\sigma_i$ and $\varphi_i$ in terms of our ennuple, viz $\sigma_i = \lambda_i\sigma_{/\lambda} + \rho_i\sigma_{/\rho} + \xi_i\sigma_{/\xi}$ and $\varphi_i = \lambda_i\varphi_{/\lambda} + \rho_i\varphi_{/\rho} + \xi_i\sigma_{/\xi}$ , then (9.8) becomes

$$(9.10) \qquad \varphi\hat{\xi}_{i;j}\hat{\xi}^j = (\varphi_{/\lambda} - \varphi\sigma_{/\lambda})\lambda_i + (\varphi_{/\rho} - \varphi\sigma_{/\rho})\rho_i \ .$$

If the equations (9.2) hold, then

$$(9.11) \qquad \hat{\xi}_{i;j}\hat{\xi}^j = 0 \ ,$$

and $\hat{\underset{\sim}{\xi}}$ defines a geodesic normal congruence. If, on the other hand, $\hat{\xi}_{i;j}\hat{\xi}^j = 0$ , then by the linear independence of $\underset{\sim}{\lambda}$ and $\underset{\sim}{\rho}$ equations (9.2) must hold. The proof is complete.

<u>**Corollary 9.2**</u>. Theorem 9.1 remains true even if $\underset{\sim}{\lambda}$ and $\underset{\sim}{\rho}$ are not canonical with respect to $\underset{\sim}{\xi}$ .

NB: in this corollary $\{\underset{\sim}{\lambda}, \underset{\sim}{\rho}, \underset{\sim}{\xi}\}$ must be orthogonal.

<u>Corollary 9.3</u>. If $\varphi = e^{\sigma}$, then equations (9.2) are satisfied.

<u>**Theorem 9.4**</u>. The differential system given by (9.2) is completely integrable.

Proof. Suppose we choose $\varphi = e^{\tau}$. Then (9.2) becomes

$$(9.12) \qquad \begin{cases} (\tau_i - \sigma_i)\lambda^i = 0 \,, \\ (\tau_i - \sigma_i)\rho^i = 0 \,, \end{cases}$$

and if $f = \tau - \sigma$, then we have

$$(9.13) \qquad \begin{cases} f_i\lambda^i = 0 \,, \\ f_i\rho^i = 0 \,. \end{cases}$$

Let $\{\underset{\sim}{\lambda}, \underset{\sim}{\rho}, \underset{\sim}{\xi}\} = \{\underset{\sim}{e}_a\}$ $a = 1, 2, 3$. Equations (9.13) may [illegible]

expressed in the form

$$(9.14) \qquad f_i e_a{}^i := f_{,a} = 0 \,,$$

where $a = 1, 2$. These equations [illegible]

only if

(9.15) [illegible]

where $a, b = 1,$ [illegible]

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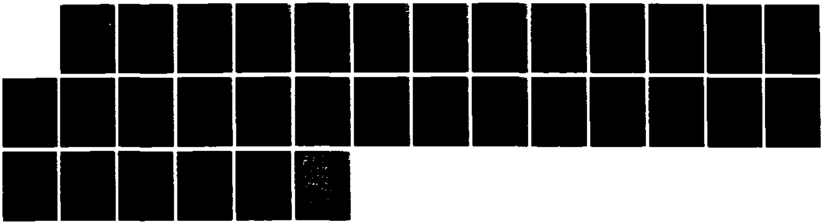

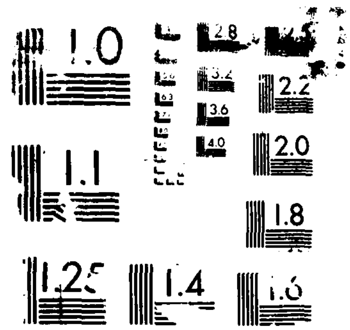
1.0
2.8
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4.0
2.0
1.1
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1.4
1.6

$$f_{/a/b} - f_{/b/a} = \sum_c (\gamma_{cab} - \gamma_{cba}) f_{/c} \,, \tag{9.16}$$

but since $f_{/c} = 0$ when $c = 1, 2$ we have

$$f_{/a/b} - f_{/b/a} = (\gamma_{3ab} - \gamma_{3ba}) f_{/3} \,, \tag{9.17}$$

and the normality of the congruence defined by $\underset{\sim}{\xi} = \underset{\sim}{e}_3$ then requires

$$f_{/a/b} - f_{/b/a} = 0 \,. \tag{9.18}$$

This completes the proof.

## §10. CONFORMAL MAPPING OF A $V_2$

In this section we consider conformally related two-dimensional spaces. We do this because much of geodesy is two-dimensional, and also to correct a formula in MARUSSI[1985].

Let $V_2$ be a two-dimensional Riemannian space with metric tensor $a_{\alpha\beta}$, Christoffel symbols $\Gamma_{\alpha\beta}{}^{\gamma}$, curvature tensor $R_{\alpha\beta\gamma\delta}$, Ricci tensor $R_{\alpha\beta}$, scalar curvature $R$, and Gaussian curvature $K$. Greek indices will, of course, range from 1 to 2.

Let $V_2$ and $\hat{V}_2$ be conformally related with conformal function $\sigma$. Then the metric tensor of $\hat{V}_2$ is given by

$$\hat{a}_{\alpha\beta} := e^{2\sigma} a_{\alpha\beta} \,, \tag{10.1}$$

the Christoffel symbols are

$$\hat{\Gamma}_{\alpha\beta}{}^{\gamma} := \Gamma_{\alpha\beta}{}^{\gamma} + \delta_{\alpha}{}^{\gamma}\sigma_{\beta} + \delta_{\beta}{}^{\gamma}\sigma_{\alpha} - a_{\alpha\beta}a^{\alpha\delta}\sigma_{\delta} \,, \tag{10.2}$$

and the curvature tensor is defined by

$$(10.3)\quad \hat{R}_{\alpha\beta\gamma\delta} := e^{2\sigma}(R_{\alpha\beta\gamma\delta} + a_{\alpha\delta}\sigma_{\beta\gamma} + a_{\beta\gamma}\sigma_{\alpha\delta} - a_{\alpha\gamma}\sigma_{\beta\delta}$$

$$- a_{\beta\delta}\sigma_{\alpha\gamma} + (a_{\alpha\delta}a_{\beta\gamma} - a_{\alpha\gamma}a_{\beta\delta})\Delta_1\sigma) ,$$

where $\sigma_{\alpha\beta} := \sigma_{\alpha,\beta} - \sigma_{\alpha}\sigma_{\beta}$ . The Ricci tensor is given by

$$(10.4)\qquad \hat{R}_{\alpha\beta} := \hat{a}^{\gamma\delta}\hat{R}_{\gamma\alpha\beta\delta} = R_{\alpha\beta} + a_{\alpha\beta}\Delta_2\sigma ,$$

and the scalar curvature is

$$(10.5)\qquad \hat{R} := e^{-2\sigma}(R + \Delta_2\sigma) .$$

Since any $V_2$ is an Einstein space we have that the Gaussian curvature of $\hat{V}_2$ is

$$(10.6)\qquad \hat{K} = e^{-2\sigma}(K - \Delta_2\sigma) .$$

Suppose that $\underset{\sim}{\lambda}$ defines a congruence of curves $\Gamma_1$ in $V_2$ . If $\lambda_\alpha$ are the components of $\underset{\sim}{\lambda}$ , then $\underset{\sim}{\rho}$ is given by

$$(10.7)\qquad \rho^{\alpha} = \epsilon^{\gamma\alpha}\lambda_{\gamma}$$

defines a congruence $\Gamma_2$ perpendicular to $\Gamma_1$ , where $\epsilon^{\gamma\alpha}$ is the contravariant Levi-Civita dualizor. See McCONNELL[1931], p. 167.

The vectors $\underset{\sim}{\lambda}$ and $\underset{\sim}{\rho}$ satisfy the **Frenet** equations McCONNELL[1931], p. 185

$$(10.8)\qquad \begin{cases} \lambda_{\alpha,\beta}\lambda^{\beta} = \kappa_g \rho^{\alpha} , \\ \rho_{\alpha,\beta}\lambda^{\beta} = -\kappa_g \lambda^{\alpha} , \end{cases}$$

where $\kappa_g$ is the geodesic curvature of $\Gamma_1$. We also have a similar pair of equations:

$$(10.9)\qquad \begin{cases} \rho_{\alpha,\beta}\rho^{\beta} = -\kappa_g^{*} \lambda_{\alpha} , \\ \lambda_{\alpha,\beta}\rho^{\beta} = \kappa_g^{*} \rho_{\alpha} , \end{cases}$$

where $\kappa_g^{*}$ is the geodesic curvature of the congruence $\Gamma_2$. We wish to exhibit similar formulas for the conformal images of $\Gamma_1$ and $\Gamma_2$ in $\hat{V}_2$.

The conformal image of $\underset{\sim}{\lambda}$ is defined by

$$(10.10)\qquad \hat{\lambda}_{\alpha} = e^{\sigma}\lambda_{\alpha} ,$$

and the conformal image of $\underset{\sim}{\rho}$ is given by

$$(10.11)\qquad \hat{\rho}_{\alpha} = e^{\sigma}\rho_{\alpha} .$$

By a straightforward computation we see that the analogous formulas to (10.8) are

$$(10.12)\qquad \begin{cases} \hat{\lambda}_{\alpha;\beta}\hat{\lambda}^{\beta} = \hat{\rho}_{\alpha} e^{-\sigma}(\kappa_g + \sigma_{/\rho}) , \\ \hat{\rho}_{\alpha;\beta}\hat{\lambda}^{\beta} = -\hat{\lambda}_{\alpha} e^{-\sigma}(\kappa_g + \sigma_{/\rho}) , \end{cases}$$

where $\sigma_{/\rho} := \sigma_{\alpha}\rho^{\alpha}$. Thus the geodesic curvature of $\hat{\Gamma}_1$ is given by

$$(10.13) \qquad \hat{\kappa}_g = e^{-\sigma}(\kappa_g + \sigma_{/\rho}) .$$

Similarly, we have formulas analogous to (10.9),

$$(10.14) \qquad \begin{cases} \hat{\rho}_{\alpha;\beta}\hat{\rho}^{\beta} = -e^{-\sigma}(\kappa_g{}^{*} + \sigma_{/\lambda})\hat{\lambda}_{\alpha} , \\ \hat{\lambda}_{\alpha;\beta}\hat{\rho}^{\beta} = e^{-\sigma}(\kappa_g{}^{*} + \sigma_{/\lambda})\hat{\rho}_{\alpha} . \end{cases}$$

Hence we have that the geodesic curvature of $\hat{\Gamma}_2$ is given by

$$\hat{\kappa}_g{}^{*} = e^{-\sigma}(\kappa_g{}^{*} + \sigma_{/\lambda}) ,$$

where $\sigma_{/\lambda} := \sigma_{\alpha}\lambda^{\alpha}$ .

Equation (10.13) is the correction to equation (4) of MARUSSI[1985] on page 150.

## §11. CRITIQUE OF HOTINE'S ARGUMENT

In this section we critically examine Hotine's argument for his conjecture that was quoted in §7.

In the un-numbered equation immediately prior to his (7.6) Hotine has an identity of the form

$$(11.1) \qquad \Delta_1(x^i, x^j) = g^{ij}$$

where the $\{x^i\}$ is a coordinate system on $V_3$ (or $E_3$) and $g^{ij}$ are the contravariant components of the metric tensor in this coordinate system. However, (11.1) is not a general tensor expression, but merely an algebraic identity that holds only at a point. To see this observe that the left-hand side of (11.1) are scalars and the right-hand side are the components of a tensor. In order to obtain the line element (7.7), Hotine has defined a conformal map of $V_3 \to \hat{V}_3$ by

$$\hat{g}_{ij} = \Delta_1(x^1, x^1) g_{ij} = g^{11} g_{ij} . \tag{11.2}$$

However, the second equality in (11.2) holds only at a point and to have a valid definition of a conformal map the conformal function must be arbitrary. Hence (11.2) represents a non-trivial specialization of the conformal function $\sigma$ in

$$\hat{g}_{ij} = e^{2\sigma} g_{ij} . \tag{11.3}$$

Furthermore, we have

$$g^{ij}{}_{,k} = 0 , \tag{11.4}$$

where "," denotes covariant derivative with respect to the Christoffel symbols of $g_{ij}$ , and thus (11.4) shows that the $\hat{g}_{ij}$ defined by (11.2) have the same Christoffel symbols as $g_{ij}$ .

The remainder of Hotine's argument is entirely based on such a specialization of the conformal function. In the conformal geometry discussed in Chapter II, $\sigma$ is always an arbitrary function. Any specialization of it generally makes it constant or forces $V_N$ or $\hat{V}_N$ to be flat. Indeed the only known $N \geq 3$ specialization of $\sigma$ is the concircular mapping defined by

$$\sigma_{i,j} - \sigma_i \sigma_j = \varphi g_{ij} , \tag{11.5}$$

but this is achieved only by the introduction of a new arbitrary function $\varphi$ .

Hotine may have been misled by two examples in McCONNELL[1931], which is one of his major references. The first of these examples is #4, p. 197, which states:

> If we choose space coordinates to be orthogonal and such that $x^3 = 0$ is the given surface and choose the u-curves on the surface to be the intersections of the surface with the $x^1$ - and $x^2$ - surfaces, then we have the following relations for all points on the surface
>
> (i) $$x_1^r = \frac{\partial x^r}{\partial u^1} := \delta_1^r \ , \ x_2^r := \frac{\partial x^r}{\partial u^2} = \delta_2^r$$
>
> (ii) $$a_{\alpha\beta} = g_{\alpha\beta},\ g_{\alpha 3} = 0;\ a^{\alpha\beta} = g^{\alpha\beta},\ g^{\alpha 3} = 0,\ g^{33} = \frac{1}{g_{33}}$$
>
> (iii) $$\xi^r = (0,0,\frac{1}{\sqrt{g_{33}}}).$$

The only thing explicit in McConnell's statement is that the $x^r$ are orthogonal <u>**curvilinear**</u> (<u>not</u> Cartesian) coordinates for $E_3$ . A similar ambiguity occurs in EISENHART[1947], p. 159.

Presumably (i) is a formal specialization of

(11.6) $$\frac{\partial x^r}{\partial x^s} = \delta_s^r$$

which is a familiar result, e.g., EISENHART[1949], p. 2, that occurs in the classical treatment of tensor transformation laws. However, (11.6) is valid only at a point -- a fact not stressed by McCONNELL[1931] but which is always implicit in the tensor transformation laws.

If we consider (i) as a system of differential equations we have

$$(11.7)\qquad \begin{aligned} x^1 &= u^1 + c^1 \\ x^2 &= u^2 + c^2 \\ x^3 &= c^3 \end{aligned}$$

where $c^1$ , $c^2$ , and $c^3$ are constants. Thus if the differential equations hold at <u>all</u> points of the surface, then the surface would be flat. This is easy to see if one examines the Lamé equations in Chapter III, §3 and the usual formula for Gaussian curvature when the surface metric is orthogonal, viz.

$$K = -\frac{1}{2\sqrt{a}}\,[(\frac{1}{\sqrt{a}}\,a_{22|1})|1 + (\frac{1}{\sqrt{a}}\,a_{11|2})|1] \;.$$

It can be arranged that (i) and hence (ii) and (iii) hold at a point of the surface. Hence this example is incorrectly stated -- the quoted results are valid only at a point, not on a surface.

McConnell's second example is #1, page 188, which is similar to equation (11.1) and states that

$$(11.8)\qquad \Delta_1(u^\alpha, u^\beta) = a^{\alpha\beta}$$

where $a^{\alpha\beta}$ are the contravariant components of the surface metric and $u^\alpha$ are coordinates on the surface. The second example essentially involves the first since by definition

$$(11.9)\quad \Delta_1(u^\alpha, u^\beta) = a^{\gamma\delta}\frac{\partial u^\alpha}{\partial u^\gamma}\frac{\partial u^\beta}{\partial u^\delta} = a^{\gamma\delta}\delta^\alpha_\gamma\delta^\beta_\delta = a^{\alpha\beta} .$$

Presumably McConnell's second example was suggested by the discussion of BIANCHI[1910], pp. 67-69, which is correct. Since this discussion is very instructive we will briefly outline it. Bianchi seeks to construct a non-singular change of variable

$$(11.10)\qquad u^\alpha \longrightarrow \bar{u}^\alpha = v^\alpha(u^1, u^2)$$

which reduces the first fundamental form $ds^2 = a_{\alpha\beta}du^\alpha du^\beta$ on a surface $V_2$, to $\bar{a}_{\alpha\beta}dv^\alpha dv^\beta$ having

$$(11.11)\qquad \bar{a}_{11} = \bar{a}_{22},\ \bar{a}_{12} = 0 .$$

Then by (11.10), the correct statement of (11.8) is

$$(11.12)\qquad \Delta_1(v^\alpha, v^\beta) = \bar{a}^{\alpha\beta} .$$

This follows by the definition of $\Delta_1$ (see Chapter II, §3)

$$(11.13)\qquad a^{\rho\sigma}\frac{\partial v^\alpha}{\partial u^\rho}\frac{\partial v^\beta}{\partial u^\sigma} = a^{\rho\sigma}\frac{\partial \bar{u}^\alpha}{\partial u^\rho}\frac{\partial \bar{u}^\beta}{\partial u^\sigma} = \bar{a}_{\alpha\beta} ,$$

which is the classical tensor transformation law. However, (11.13) holds not only at a point but in a coordinate chart of the point.

Writing $\xi := v^1$ and $\eta := v^2$, (11.12) becomes

$$(11.14)\qquad \Delta_1\xi = \bar{a}^{11},\ \Delta_1(\xi,\eta) = \bar{a}^{12},\ \Delta_1\eta = \bar{a}^{22} .$$

But since

$$(11.15) \qquad (\Delta_1\xi)(\Delta_1\eta) - \Delta_1(\xi,\eta)^2 = \frac{1}{\bar{a}} = \frac{1}{a} \left| \frac{\partial(\xi,\eta)}{\partial(u^1,u^2)} \right|^2 = 0 ,$$

where $a := |a_{\alpha\beta}|$ , $\bar{a} := |\bar{a}_{\alpha\beta}|$ , it follows that

$$(11.16) \quad \bar{a}_{11} = \bar{a}\Delta_1\xi, \ \bar{a}_{12} = -\bar{a}\Delta_1(\xi,\eta), \ a_{22} = \bar{a}\Delta_1\eta .$$

Hence the condition, $\bar{a}_{12} = 0$ , that the new coordinate lines $\xi$ = constant, $\eta$ = constant on $V_2$ be orthogonal is that

$$(11.17) \qquad \Delta_1(\xi,\eta) = 0 ,$$

while $\bar{a}_{11} = \bar{a}_{22}$ requires that

$$(11.18) \qquad \Delta_1\xi = \Delta_1\eta .$$

Explicitly expanding these expressions, and solving for the partial derivatives $\eta_\alpha$ , gives

$$(11.19) \qquad \begin{cases} \eta_1 = \dfrac{a_{12}\xi_2 - a_{11}\xi_1}{\sqrt{a}} , \\ \eta_2 = \dfrac{a_{22}\xi_1 - a_{12}\xi_2}{\sqrt{a}} . \end{cases}$$

The integrability conditions of these equations require that $\xi$ be a (real) solution of

$$(11.20) \qquad \Delta_2\xi = 0 .$$

A similar procedure for the partial derivatives $\xi_\alpha$ gives

$$\xi_1 = \frac{a_{11}\eta_2 - a_{12}\eta_1}{\sqrt{a}} ,$$

(11.21)

$$\xi_2 = \frac{a_{12}\eta_1 - a_{22}\eta_1}{\sqrt{a}} .$$

So that $\eta$ must be a (real) solution of

(11.22) $$\Delta_2\eta = 0 .$$

The systems (11.19) and (11.21) are called **Beltrami systems** and they are the **Cauchy-Riemann equations** on the **curved** surface $V_2$. Likewise (11.20) and (11.22) are Laplace's equations on $V_2$.

Thus the required reduction of $a_{\alpha\beta}du^{\alpha}du^{\beta}$ to $\lambda(\xi,\eta)\{d\xi^2 + d\eta^2\}$ is equivalent to determining solutions of a pair of Beltrami systems. This problem is now other than the construction of an isothermal coordinate system on $V_2$, and in effect explains our comment on page 26 that any $V_2$ is conformal to any other $V_2$. There are an infinite number of systems of isothermal coordinates, each system corresponding to an analytic function of the complex variable $u^1 + iu^2$. More precisely stated, if the coordinate lines $u^1$ = constant, $u^2$ = constant then **all** other isothermal systems are given by the equations

$$\text{Re}\{f(u^1 + iu^2)\} = \text{constant},$$

(11.23)

$$\text{Im}\{f(u^1 + iu^2)\} = \text{constant},$$

where $f$ is an analytic function of $u^1 + iu^2$. Thus corresponding to each isothermal system there is a conformal mapping

$V_2 \to \hat{V}_2 = E_2$ .

All of Bianchi's work is two-dimensional; however, Hotine's argument requires a more complicated construction in three dimensions, and his proof involves a serious omission. In effect he specializes the conformal function

$$\varphi^2 = \Lambda_1 \phi = e^{2\sigma} \tag{11.24}$$

to obtain the line element (7.3). This is employed to map the system of surfaces $\phi$ = constant into a system of geodesic parallel surfaces. Since the function $\phi$ is not truly arbitrary, it remains to be shown that such a specialization is valid. In order to complete Hotine's argument, it would be necessary to prove that his specialization was admissible. In view of the previously indicated error there seems little reason to attempt such a proof. Moreover, it seems highly unlikely that such a proof could be done since the requirement that $\sigma$ map a system of geodesically parallel surfaces into a system of geodescially parallel surfaces forces $\sigma$ to reduce to a constant, i.e., a homothety. Moreover, Bianchi's reduction of $a_{\alpha\beta} du^\alpha du^\beta$ to isothermal form was possible only by the existence of harmonic conjugate functions, and is a direct consequence of the plentiful supply of analytic functions of a complex variable. In Hotine's case he requires a more complicated specialization of $g_{ij} dx^i dx^j$ to the form (7.3). However, in three dimensions the situation is significantly different. The three-dimensional analogue of the Beltrami systems (see HEDRICK-INGOLD[1925]) does not reduce to Cauchy-Riemann-like systems and the supply of analytic functions --

hence conformal maps -- is quite limited. This, of course, was predicted by the Liouville Theorem in Chapter III, §3. For this reason it seems unlikely that Hotine's argument could be completed.

Finally we can also consider Hotine's conjecture strictly from the viewpoint of partial differential equations. let $\xi : V_3 \to \mathbb{R}$ be a **given** function with non-vanishing gradient. From partial differential equations we know that the system

$$(11.25) \qquad \begin{cases} g^{ij}\xi_i\eta_j := \Delta_1(\xi,\eta) = 0 \\ g^{ij}\xi_i\zeta_j := \Delta_1(\xi,\zeta) = 0 \end{cases}$$

has solutions $\eta,\zeta : V_3 \to \mathbb{R}$ . If in addition we require that

$$(11.26) \qquad g^{ij}\zeta_j = \Delta_1(\eta,\zeta) = 0 ,$$

that is, $\{\xi,\eta,\zeta\}$ define a triply orthogonal system, then the system of partial differential equations given by (11.25) and (11.26) is over- determined and need not have a solution. Another equation on $\xi$ is needed to ensure a solution and this new equation is precisely the Cayley-Darboux equation of Chapter III. In fact, the Cayley-Darboux equation is derived from a system of partial differential equations analogous to (11.25) and (11.26).

It is now easy to see the connection between the generalized Dupin theorem and the Cayley-Darboux equation. If we have a system of surfaces defined by $\xi$ = constant, then, it is always possible to find another system orthogonal to it. However, the generalized Dupin theorem states that the existence of a third system orthogonal to

**both** of the others requires the original two systems to intersect in lines of curvature. Therefore, we may restate the Cayley-Darboux theorem in the following manner.

If $\xi : V_3 \to \mathbb{R}$ defines a system of surfaces, then in order for there to exist another system of surfaces orthogonal to that defined by $\xi$ and such that the two systems intersect in lines of curvature, $\xi$ must satisfy the Cayley-Darboux equation.

In conclusion we have indicated three major reasons why Hotine's argument is seriously flawed:

(i) it involves a choice of coordinates, and simplification of original metric which is valid only at a point;

(ii) it employs a specialization of the conformal mapping function, but does not establish the admissibility of this specialization;

(iii) it does not verify that the geoidal surface given by $\phi$ = constant satisfies the Cayley-Darboux equation.

Any one of these reasons would be non-trivial to rectify, and taken together we feel they show that Hotine's argument is fatally flawed.

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## Appendix - EXAMPLE OF THE CALCULATION OF CANONICAL CONGRUENCES

Let $V_3 = \mathbb{R}^3 = E_3$ and set

$$f(x^1, x^2, x^3) = (x^1)^2 + (x^2)^2 \tag{A.1}$$

and for the purpose of convenience, rewrite (A.1) as $f = x_1{}^2 + x_2{}^2$. Then

$$\underset{\sim}{\nabla} f = 2(x_1, x_2, 0) \tag{A.2}$$

hence the unit normal to the surfaces $f$ = constant is

$$\underset{\sim}{\xi} = \frac{1}{2f^{1/2}} \underset{\sim}{\nabla} f , \tag{A.3}$$

and (5.8) becomes

$$\begin{vmatrix} \dfrac{x_2{}^2}{2f^{3/2}} - \omega & -\dfrac{1}{2f^{3/2}} \, x_1 x_2 & 0 & \dfrac{x_1}{2f^{1/2}} \\ -\dfrac{1}{2f^{3/2}} \, x_1 x_2 & \dfrac{x_1{}^2}{2f^{3/2}} - \omega & 0 & \dfrac{x_2}{2f^{1/2}} \\ 0 & 0 & -\omega & 0 \\ \dfrac{x_1}{2f^{1/2}} & \dfrac{x_2}{2f^{1/2}} & 0 & 0 \end{vmatrix} = 0. \tag{A4}$$

Evaluation of this determinant yields

$$\omega \left[ \frac{\omega}{4} - \frac{1}{8f^{1/2}} \right] = 0 \tag{A.5}$$

which implies that

(A.6) $$\omega_1 = 0 \quad \text{or} \quad \omega_2 = \frac{1}{2}f^{-1/2} .$$

Using the second equation of (5.7) the vector corresponding to $\omega_1 = 0$ is given by

(A.7) $$X_{ij}\lambda^i\lambda^j = 0$$

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}^T \begin{bmatrix} x_2^{\,2} & -x_1x_2 & 0 \\ -x_1x_2 & x_1^{\,2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = 0$$

and the vector corresponding to $\omega_1 = \frac{1}{2} f^{-1/2}$

(A.8) $$X_{ij}\rho^i\rho^j = \frac{1}{2f^{1/2}}$$

or

$$\frac{1}{2f^{3/2}} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix}^T \begin{bmatrix} x_2^{\,2} & -x_1x_2 & 0 \\ -x_1x_2 & x_1^{\,2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix} = -\frac{1}{2f^{1/2}} .$$

If we make the change of variables

(A.9) $$x_1 = r\cos\theta , \quad x_2 = r\sin\theta , \quad x_3 = x_3$$

we obtain

(A.10) $$\underset{\sim}{\lambda} = (0,0,1) ,$$

(A.11) $$\underset{\sim}{\rho} = (-\sin\theta, \cos\theta, 0) \; ,$$

and

(A.12) $$\underset{\sim}{\xi} = (\cos\theta, \sin\theta, 0)$$

which are the well-known basis vectors for cylindrical coordinates.

SUPPLEMENT (by J.D. ZUND)

## §1. Introduction

The work in Dr. Moore's dissertation strongly suggests that Hotine's conjecture is highly implausible. It shows that crucial steps in his arguments are fatally flawed, but it does not prove that the conjecture is false. Without using Hotine's equations we cannot obtain his results, and we were unable to derive his equations by other means. Our goal was then to find a decisive part of his argument which was wrong. We succeeded in doing this about a week after the dissertation was submitted to the examining committee and the Graduate School of New Mexico State University. In this supplement I will describe this conclusive error in more detail than was possible in our joint paper "Hotine's Conjecture and Differential Geodesy," which will be published in Bulletin Geodesique.

As mentioned in our introduction to this research report, the major oversight on Hotine's part was his failure to employ the formalism of Ricci rotation coefficients. In Section §2 we will translate Hotine's Cayley-Darboux equation into the language of rotation coefficients. In Section §3 it will be shown that his Cayley-Darboux equation is not equivalent to the true Cayley-Darboux equation and moreover neither of these equations is an identity. Finally, in Section §4 we will discuss the physical ramifications of this result and why MARUSSI [1985] (page 133) was correct in his 1967 statement that "there is no possibility of reducing the study of the earth's potential field onto a triply-orthogonal system."

## §2. Hotine's Cayley-Darboux Equation

In this section we will reformulate Hotine's Cayley-Darboux equation in the language of rotation coefficients. Before doing this, a few general comments on notation are required. First, our paper "Hotine's Conjecture and

Differential Geodesy" for Bulletin Geodesique was essentially written in Hotine's notation which differs slightly from that employed in Dr. Moore's dissertation. The main differences are that Hotine did not employ a comma to denote covariant derivatives, and we indicate partial derivatives by a stroke "|". Moreover, Hotine's conformal function $m^2$ appears in the dissertation as $e^{2\nu}$, and the triad $\{\lambda_a{}^r\}$ in our paper is denoted in the dissertation by $\{e_a{}^i\}$. Since this supplement is intended to be read in conjunction with our paper, it will be written in Hotine's notation. Additional differences in notation will be noted where necessary.

Second, references to equations appearing in the dissertation will include a page citation. References to our Bulletin Geodesique paper are to equations appearing in its appendix and will include the prefix "A" in the equation number. All numbered equations in this supplement will include the prefix "S" in the equation number. Finally, references in the supplement will always refer to those listed in the bibliography of the dissertation.

In HOTINE [1966b] (page 202), and in HOTINE [1969] (page 114), Hotine announced two forms of the Cayley-Darboux equation: a *surface equation*

$$(\frac{1}{n})_{\alpha\beta}\lambda^\alpha\mu^\beta = 0 \qquad (\alpha,\ \beta = 1,\ 2) \tag{S.1}$$

and a *space equation*

$$(\frac{1}{n})_{rs}\lambda^r\mu^s = 0 \qquad (r,\ s = 1,\ 2,\ 3)\ . \tag{S.2}$$

In both these equations the vectors $\lambda^\alpha$, $\mu^\beta$ and $\lambda^r$, $\mu^s$ respectively are principal directions. The factor $n$ appears in the basic gradient equation:

$$N_r = n\nu_r\ , \tag{S.3}$$

and in this equation $N$ is to be identified with the geopotential of the rotating Earth; each N-surface $N(x^1,x^2,x^3)$ = constant is an equipotential surface; and $\nu_r$ is the unit normal to the N-surfaces. Hence physically n corresponds to the magnitude of the gravitational acceleration. In the dissertation, (S.3) is written with N, n, $\nu_r$ replaced by $\phi$, $\varphi$, $\xi_i$ (see (7.1) page 75).

Strictly speaking, Hotine's derivation of both (S.1) and (S.2) are rather dubious. In HOTINE [1969] (pages 113-114) few details are given and we suspect that the missing steps include use of equations which are valid only at a point (see the dissertation pages 90-91). However, the real issue is not how he derived these equations, but the fact that he regarded them -- in particular (S.2) -- as the genuine Cayley-Darboux equations.

In order to translate (S.2) into rotation coefficient language, we take the unit vector $\nu_r$ to be the third vector of our triad $\{\lambda_a{}^r\}$, with the unit vectors $\lambda_r$ and $\mu_r$ being the first two vectors of the triad. These vectors, or equivalently the congruences $\Gamma_1$ and $\Gamma_2$ having them as tangent vectors, are assumed to be canonical with respect to $\nu_r$. Hotine did not use this terminology, but this important property is implicit in his construction. This notion is fully discussed on pages 65-68 of the dissertation. The resulting triad is denoted as $\{\lambda_a{}^r\}$ and explicitly it is

$$\{\lambda^r, \mu^r, \nu^r\} .$$

We now proceed to translate (S.3) into rotation coefficients. First we covariantly differentiate it to obtain

(S.4) $$N_{rs} = n_s\nu_r + n\nu_{rs} ,$$

and similarly

(S.5) $$N_{sr} = n_r\nu_s + n\nu_{sr} .$$

But since $N_r$ is a gradient, both these expressions are equal, viz

$N_{rs} = N_{sr}$! Using (S.4) and (S.5) we obtain

$$N_{rs}\nu^r\mu^s = (n_s\nu_r + n\nu_{rs})\nu^r\mu^s = n_{/2} ,$$

$$N_{sr}\nu^r\mu^s = (n_r\nu_s + n\nu_{sr})\nu^r\mu^s = n\gamma_{323} ,$$

since $\nu_r$ is a unit vector, i.e., $\nu_r\nu^r = 1$, we have $\nu_{rs}\nu^r = 0$. Thus we obtain

$$n_{/2} = n\gamma_{323} \tag{S.6}$$

where $n_{/2} = n_s\mu^s$ and "/" is defined as in (A.2), or in the dissertation on pages 54 and 58. Repeating the same procedure for $N_{rs}\nu^r\lambda^s$ and $N_{sr}\nu^r\lambda^s$ yields the corresponding equation

$$n_{/1} = n\gamma_{313} \tag{S.7}$$

where $n_{/1} = n_s\lambda^s$.

It is convenient now to temporarily write $\psi \equiv \frac{1}{n}$, so (S.2) becomes

$$\psi_{rs}\lambda^r\mu^s = 0 . \tag{S.8}$$

To convert this into rotation coefficients, we compute the derivatives

$$\psi_r = -\frac{1}{n^2} n_r ,$$

$$\psi_{rs} = -\frac{1}{n^3} (nn_{rs} - 2n_r n_s) ,$$

and substitute them into (S.8). This gives the equation

$$nn_{rs}\lambda^r\mu^s - 2n_r n_s\lambda^r\mu^s = 0 . \tag{S.9}$$

On the other hand, by definition

$$n_{/1/2} \equiv (n_r\lambda^r)_s\mu^s ,$$

i.e.,

$$n_{/1/2} = (n_{rs}\lambda^r \mu^s + n_r \lambda^r{}_s \mu^s) \; .$$

However, we have

$$\lambda^r{}_s \mu^s = \gamma_{122}\mu^r + \gamma_{132}\nu^r$$

and since, by hypothesis $\Gamma_3$ is normal, $\gamma_{312} = \gamma_{321}$ (see (5.2) page 66) and canonical $\gamma_{132} = -\gamma_{312} = 0$ (see (5.11) page 69), this reduces to merely

$$\lambda^r{}_s \mu^s = \gamma_{122}\mu^r \; .$$

Thus, we obtain

$$n_{/1/2} = n_{rs}\lambda^r \mu^s + \gamma_{122} n_{/2} \; ,$$

i.e.,

$$n_{rs}\lambda^r \mu^s = n_{/1/2} - \gamma_{122} n_{/2} \; . \tag{S.10}$$

Now using (S.7) and differentiating, we see that

$$n_{/1/2} = \gamma_{313} n_{/2} + n\gamma_{313/2}$$

and by (S.6) this yields

$$n_{/1/2} = n(\gamma_{323}\gamma_{313} + \gamma_{313/2}) \; .$$

Thus, combining these with (S.10) we have finally

$$n_{rs}\lambda^r \mu^s = n(\gamma_{323}\gamma_{313} + \gamma_{313/2} - \gamma_{323}\gamma_{122}) \; .$$

If this is substituted into (S.9) we obtain

$$n^2(\gamma_{323}\gamma_{313} + \gamma_{313/2} - \gamma_{323}\gamma_{122} - 2\gamma_{323}\gamma_{313}) = 0$$

which simplifies to

$$\gamma_{313/2} - \gamma_{323}\gamma_{313} - \gamma_{323}\gamma_{122} = 0 \tag{S.11}$$

and this is Hotine's Cayley-Darboux equation.

## §3. The Cayley-Darboux Equation

We now establish that Hotine's Cayley-Darboux equation is not equivalent to the true Cayley-Darboux equation, and that neither this equation, nor (S.11), is an identity. This conclusively refutes the conjecture.

The first part is easy. In the case of a triply-orthogonal system of surfaces, the respective surfaces pairwise interest in the normal canonical congruences $\Gamma_1$, $\Gamma_2$, $\Gamma_3$ as described in the triad scheme of §2. Thus, when $\gamma_{312} = \gamma_{321} = 0$ (the condition for the normal congruences $\Gamma_1$ and $\Gamma_2$ to be canonical with respect to $\Gamma_3$), as shown in the dissertation on page 74, the Cayley-Darboux equation assumes the simple form

$$\gamma_{123} = 0 \ . \tag{S.12}$$

It is now obvious that this does not have the same structure as Hotine's equation (S.11).

Hotine claimed that his equation was an identity by virtue of the Lamé equations (page 25) which express the flatness of Euclidean 3-space. In terms of rotation coefficients these are expressed by the six equations

$$R_{abcd} = 0 \ , \tag{S.13}$$

viz

$$R_{1212} = 0, \quad R_{1213} = 0, \quad R_{1223} = 0 \ ,$$

$$R_{1313} = 0, \quad R_{1323} = 0, \quad R_{2323} = 0 \ ,$$

where

$$R_{abcd} \equiv \gamma_{abc/d} - \gamma_{abd/c}$$

$$(S.14) \qquad + \sum_f (\gamma_{fad}\gamma_{fbc} - \gamma_{fac}\gamma_{fbd})$$

$$+ \sum_f \gamma_{abf}(\gamma_{fcd} - \gamma_{fdc}) \; .$$

By an inspection of (S.14) it is clear that only *one* of the Lamé equations has a form similar to (S.11). This is $R_{1323}$, which by skew-symmetries is $R_{3132}$, and

$$R_{3132} = \gamma_{313/2} - \gamma_{312/3}$$

$$+ \sum_f (\gamma_{f32}\gamma_{f13} - \gamma_{f33}\gamma_{f12}$$

$$+ \sum_f \gamma_{31f}(\gamma_{f32} - \gamma_{f23}) \; .$$

Expanding this expression using the normal and canonical condition $\gamma_{312} = \gamma_{321} = 0$, we obtain

$$(S.15) \quad 0 = \underline{\gamma_{313/2}} - \underline{\gamma_{313}\gamma_{323}} - \underline{\gamma_{323}\gamma_{122}} - \gamma_{123}(\gamma_{311} - \gamma_{322})$$

where the underlined terms are precisely those appearing in (S.11). This shows that Hotine's equation is merely *a piece of a lame equation*, and (S.11) does *not imply that* $\gamma_{123} = 0$ which is the true Cayley-Darboux equation. Moreover, (S.15) shows that $\gamma_{123}$ need not be zero, hence the Cayley-Darboux equation is certainly not an identity!

## §4. Physical Consequences

In §3 we showed that neither Hotine's Cayley-Darboux equation or the true Cayley-Darboux equation is an identity, i.e., a consequence of the Lamé equations for the flatness of $E_3$. Thus, in order for $E_3$ to admit a triply-orthogonal system of coordinates the true Cayley-Darboux must be imposed as an additional condition. This is presumably the reason for the cryptic comment of Marussi quoted at the end of the introduction to this supplement.

The reasons for such a conclusion are now obvious, from the complicated nature of the Cayley-Darboux equation (recall §6 of Chapter IV, pages 73-74). One can argue both on mathematical and physical grounds, and in fact such reasons are not truly independent.

Suppose one considers the simple case of the Earth rotating with a uniform angular velocity $\tilde{\omega}$ and having $N$ as its geopotential function. Then one has the Newtonian equation

$$\Delta N = -2\tilde{\omega}^2 \tag{S.16}$$

(where $\Delta$ is the 3-dimensional Laplacian), and if the function $N$ is to define a N-surface of a triply-orthogonal system of surfaces, then it must also satisfy the Cayley-Darboux equation

$$\aleph N = 0 \; . \tag{S.17}$$

Equation (S.17) is a convenient way of writing the Cayley-Darboux equation (it was not used in Chapter III, or ZUND/MOORE [1986], but it was introduced in our Bulletin Géodésique paper). In effect, $\aleph$ might be called the Cayley-Darboux operator which as we know is a complicated non-linear third order non-linear partial differential operator. A tensor expression for (S.17) was given in ZUND/MOORE [1986], i.e.,

$$\epsilon_{ijk}\epsilon_{\ell mn}\epsilon_{pqr}N_{i\ell}(N_s N_{jps} - 2N_{js}N_{ps})\delta_{mq}N_k N_n N_r = 0 \; , \tag{S.18}$$

where $\epsilon_{ijk}$ is the Levi-Civita permutation tensor and the subscripts on the function $N$ denote partial derivatives.

Mathematically for a triply-orthogonal system one is faced with solving the combined system of equations (S.16) and (S.17). This system is rather unspeakable, and there is no reason to suspect that these two equations are even consistent. Even worse are the technical questions of the existence and uniqueness of such a combined linear and non-linear set of equations. Virtually nothing is known in this respect! From the underlying physical situation, one would like to regard (S.16) as the basic equation for $N$ with (S.17) being some kind of geometric constraint on the function N. However, mathematically these roles might be interchanged! Even if all these questions are successfully resolved and (S.17) is regarded as a geometric constraint on the problem, there is no guarantee that it will not exclude all the interesting solutions of (S.16). Thus, mathematically the combined system is a horrendous problem.

Physically the situation is much simpler. It is clear that (S.16) is the significant equation and that it is sufficient to determine the Newtonian geopotential function N. The theory is complete and requires only *one equation*, (S.16), and there is really no need for (S.17) in the physics of the Earth's gravity field. Barring an unlikely physical interpretation for (S.17) -- which would make potential theory into a non-linear theory (a non-Newtonian theory) -- this equation is nonsense. Its only purpose is to produce a triply-orthogonal system of surfaces and a 'nice' coordinate system. However, its complicated nature -- both mathematically and physically -- since it could exclude physically meaningful solutions of (S.16) -- suggests that such coordinate systems need not exist. We have no doubt that if Hotine had known that the correct Cayley-Darboux was not an identity, he would have not proposed his conjecture. Stated succinctly, (S.17) is not a physical equation, and it does not fit into the structure of Newtonian potential theory and should be discarded.

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